Real Analysis

April 2006 **Ph.D. Qualifying Exam** R. Nagisetty & Denis White

Answer any six questions.

1. For $n = 1, 2, 3\ldots$, let $f_n(x) = (1/n)\arctan(n^2x^2)$ for x in $\mathbb{R}$. Show that f_n converges uniformly on the entire real line and moreover that the sequence f_n' of derivatives converges pointwise on all of $\mathbb{R}$. Show also that f_n' does not converge uniformly on any interval containing the origin.

2. Prove or disprove the following. Suppose $(a_n)_{n\geq 1}$ is a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$ and such that the partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

are bounded for every positive integer N. Then $\sum_{n=1}^{\infty} a_n$ converges.

3. (a) State the Stone Weierstrass Theorem.

   (b) Let X and Y be compact Hausdorff spaces and let $C(X)$ denote the space of continuous functions defined on X. For $f \in C(X)$ and $g \in C(Y)$ define the function $f \otimes g$ on $X \times Y$ by $f \otimes g(x, y) = f(x)g(y)$. Show that every continuous function defined on $X \times Y$ can be approximated uniformly by a finite sum $\sum_{i=1}^{N} f_i \otimes g_i$ where $f_i \in C(X)$ and $g_i \in C(Y)$.

4. Suppose that $f \in L^1(\mathbb{R})$ and g, defined by $g(x) = xf(x)$ is also in $L^1(\mathbb{R})$. Show that $\hat{f}$, defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx,$$

for $\xi \in \mathbb{R}$ is continuously differentiable and its derivative $\hat{f}'(\xi)$ is bounded.

5. (a) State the definition of equicontinuous for a set of continuous functions.

   (b) State the Arzela-Ascoli Theorem.

   (c) Suppose that K is a continuous real valued function defined on a square $[a, b] \times [a, b]$ and define T on the space $C[a, b]$ of continuous functions defined on $[a, b]$ by

$$Tf(x) = \int_a^b K(x, y)f(y)\, dy$$

   Show that the image under T of a bounded set in $C[a, b]$ has compact closure in $C[a, b]$.

6. Suppose that $f \in L^1(\mathbb{R})$. Show that, for every $\epsilon > 0$ there is $\delta > 0$ so that, if a Borel set E has Lebesgue measure at most δ then

$$|\int_E f(x)\, dx| < \epsilon$$

7. Let $f(x) = \ln(1 + x)$. Derive the MacLaurin series for $f(x)$ and show that it converges back to $f(x)$ in the interval $(-1, 1)$.

8. Define a convex function on the real line and show that it is differentiable at all but a countable set of points.

9. Let $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ be a recurrence relation with $s_1 > 0$. Show that the sequence $\{s_n\}$ converges.[Show that it is monotone and bounded.]