University of Toledo Algebra Ph.D. Qualifying Exam
April 21, 2007

Instructions: The exam is divided into three sections. Please choose exactly three problems from each section. Clearly indicate which three you would like graded. You have three hours.

1. Section I

(1) (a) Find the Sylow-3 subgroups of the symmetric group S_4.
(b) Let f be an automorphism of S_4. Show that f permutes the Sylow-3 subgroups and that if f fixes them all then f is the trivial automorphism. Conclude that $|\operatorname{Aut} S_4| \leq 24$.
(c) Show that S_4 has 24 inner automorphisms, and thus $|\operatorname{Aut} S_4| \cong S_4$ and S_4 has no outer automorphisms.

(2) Let G be a finite group, $H \leq G$, $N \trianglelefteq G$. Prove that if $|H|$ and $|G : N|$ are relatively prime then $H \leq N$.

(3) Let p and q be distinct primes and suppose that G is a finite group having exactly $p + 1$ Sylow p-subgroups and $q + 1$ Sylow q-subgroups. Prove that there exist $P \in Syl_p(G)$ and $Q \in Syl_q(G)$ such that the subgroup generated by P and Q is $PQ = P \times Q$.

(4) Let x and y be elements of a finite p-group P and let $z = [x, y]$ be the commutator $x^{-1}y^{-1}xy$ of x and y. suppose that x lies in every normal subgroup of P that contains z. Prove that $x = 1$.

(5) Let G be a group and let N be a normal subgroup of G.

(a) If G/N is a *free* group, prove that there is a subgroup K of G such that $G = NK$ and $N \cap K = 1$.
(b) Show that the conclusion in part (a) is false if G/N is not assumed to be free.

2. Section II

(6) Prove that the group of all automorphisms of the field $\mathbb{R}$ of real numbers is trivial.

(7) Determine the Galois group of $f(x) = x^4 - 2 \in \mathbb{Q}[x]$. Illustrate explicitly the lattice of subgroups and the corresponding lattice of subfields under the fundamental theorem of Galois Theory.

(8) We say a field extension K/F is *cyclic* if it is Galois and the Galois group is cyclic.

(a) Let F be a field of characteristic 0 and assume that K/F is cyclic of degree $|K : F| = n$. If d is any divisor of n, show that there is a unique intermediate field L such that L/F is cyclic of degree d.

(b) Assume (a special case of) Dedekind's theorem that, for any natural number n, there are infinitely many primes of the form $kn+1, k \in \mathbb{Z}$. Prove that for any natural number n, there is an extension of the field $\mathbb{Q}$ of rational numbers that is cyclic of degree n.

(9) Let F and K be fields with $F \subseteq K$ and assume that the extension K/F is algebraic. If $\sigma : K \to K$ is a ring homomorphism that fixes each element of F, prove that σ is an F-isomorphism.

(10) Let f be an irreducible polynomial of degree 6 over a field F. Let K be an extension field of F with $|K : F| = 2$. If f is *reducible* over K, prove that it is the product of two irreducible cubic polynomials over K.

3. Section III

(11) Let R be a commutative ring with identity and P a prime ideal.
  (a) Describe the construction of the localization of R at P, denoted R_P.
  (b) Prove there is a 1-1 correspondence between prime ideals of R which are contained in P and prime ideals of R_P.
  (c) Prove that under this correspondence the ideal P corresponds to the unique maximal ideal in R_P.
  (d) Prove this maximal ideal is exactly the set of non-units in R_P.

(12) Let I be a principal ideal in an integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements.

(13) (a) Let F be a field and let A be an $n \times n$ matrix with entries in F. State a necessary and sufficient condition on the minimal polynomial of A for A to be diagonalizable over F.

(b) Let $F = \mathbb{C}$ be the field of complex numbers. If A satisfies the equation $A^3 = -A$, show that A is diagonalizable over $\mathbb{C}$.

(c) Let $F = \mathbb{R}$ be the field of real numbers. Given that A satisfies the equation $A^3 = -A$ and given that A is diagonalizable over $\mathbb{R}$, what is the strongest conclusion that can be drawn about A?

(14) Let F be a field. Construct, up to similarity, all linear transformations $T : F^6 \to F^6$ with minimal polynomial $m_T(x) = (x-5)^2(x-6)^2$,

(15) (a) Let R be a ring and M be an R-module. What does it mean for M to be a *free* R-module?

(b) Let $\mathbb{Z}[\frac{1}{2}]$ denote the subring of Q generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}[\frac{1}{2}]$ is a free $\mathbb{Z}$-module.