Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. In each part choose four problems. Mark clearly the problems you choose and show the details of your work.

Part I: Ordinary Differential Equations

1. Consider the function sequence given by $y_0(t) = 0$ and

$$y_{k+1}(t) = 2 + \int_0^t \cos^2 s \, y_k(s) ds, \quad k = 0, 1, \ldots$$

(a) Prove that the seqeunce converges unniformly on any closed interval $[-N, N]$.

(d) Find the limit function of the sequence.

2. Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval where $p(x), q(x)$ are continuous.

1) Let $y(x)$ be a non-zero solution. Prove that all the roots of $y(x)$ are isolated.

2) Let $y_1(x), y_2(x)$ be two linearly independent solutions. Prove that between any two roots of $y_1(x)$ there is exactly one root of $y_2(x)$.

3. Consider the nonlinear DE $\ddot{x} + 4\sin(x) = 0$.

(a) Sketch the trajectories in the phase plane.

(b) For the solution of the differential equation with the initial value conditions $x(0) = \pi/6$, $\dot{x}(0) = 0$, explain why the solution is periodic and find an integral expression for the period.

4. Consider the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2\pi) + y'(2\pi) = 0.$$

(a) Find all eigenfunctions and eigenvalues $(y_n(x), \lambda_n)$. Be sure to check the possibilities $\lambda_n \leq 0$.

(b) Show that the set of eigenfunctions form an orthogonal set of functions in $L^2[0, \pi]$.

(c) Solve for the λ_n graphically and make a good asymptotic estimate for λ_n.

5. (a) Find the fundamental solution matrix for the linear system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

(b) Solve the initial value problem.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^t, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

6. Suppose a non-negative function $\sigma(t)$ satisfies $\sigma(0) = 0$ and

$$|\sigma'(t)| \leq 2\sigma(t).$$

Prove that $\sigma(t) \equiv 0$.

Part II: Partial Differential Equations

1. (Poisson's Formula on a Disk) Let $f(\theta)$ be a continuous and 2π-periodic function with Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta).$$

Let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k(a_k \cos k\theta + b_k \sin k\theta).$$

(a) Prove that the series for $u(r,\theta)$ converges uniformly on any disk $B_R = \{(r,\theta) \mid 0 \le r \le R\}$ with $R < 1$.

(b) Show how to rewrite the series for $u(r,\theta)$ in the form

$$u(r,\theta) = \int_0^{2\pi} f(\phi)P(r,\theta-\phi)d\phi$$

where P is the Poisson kernel satisfying

$$P(r,\phi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\phi + r^2}.$$

(c) Prove that $\lim_{r\to 1^-} u(r,\theta) = f(\theta)$ uniformly.

2. Consider the Fourier series solution to the heat equation with an initial condition and a boundary condition

$$\begin{cases} u_t = u_{xx}, & \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = f(x), & \\ u(0,t) = u(\pi,t) = 0. & \end{cases}$$

(a) Derive the formal solution $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t}\sin kx$ where the b_k are the coefficients of the Fourier series $\sum_{k=1}^{\infty} b_k \sin kx$ for the continuous function $f(x)$.

(b) Show that for every $\delta > 0$ this solution series converges uniformly in the region $0 \le x \le \pi, t \ge \delta$. Also show the same convergence for the corresponding series for u_t, u_x, u_{xx}.

(c) Show that $\lim_{t\to 0^+} u(x,t) = f(x)$ in the L^2 norm.

3. Find a solution to the initial-boundary value problem for the heat equation

$$\begin{aligned} u_t(x,t) &= 4u_{xx}(x,t) \text{ on } 0 < x < \pi,\ t > 0 \\ u(0,t) &= 0 \text{ for } t > 0 \\ u_x(\pi,t) &= 0 \text{ for } t > 0 \\ u(x,0) &= 2\sin(\frac{x}{2}) + 5\sin(\frac{3x}{2}) \text{ for } 0 < x < \pi. \end{aligned}$$

4. Let $u(x)$ be an entire harmonic function. Suppose that $u(x) \geq -|x| + 1$ for all x. Prove that $u(x)$ is a linear function.

5. Let $B^+ = \{(x,y) \mid x^2 + y^2 < 1, y > 0\}$ be the open half disk. Suppose $u(x,y) \in C^2(B^+) \cap C^1(\bar{B^+})$ satisfies $\Delta u = u_{xx} + u_{yy} = 0$ in B^+ and $u_y(x,0) = 0$. Prove that the extension $u(x,y)$ to B defined as follows is a harmonic function on all B.

$$u(x,y) = \begin{cases} u(x,y) & \text{if } y \geq 0; \\ u(x,-y) & \text{if } y < 0. \end{cases}$$

6. Solve the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= 9u_{xx}(x,t) \text{ on } 0 < x < \infty,\ t > 0 \\ u(0,t) &= 0 \text{ for } t > 0, \\ u(x,0) &= 3\sin(x) - \frac{3}{2}\sin(2x) \text{ for } 0 < x < \infty, \\ u_t(x,0) &= \sin(2x) \text{ for } 0 < x < \infty. \end{aligned}$$