University of Toledo Department of Mathematics
Ph.D. Qualifying Exam in Algebra
April 19, 2008

Instructions: Please do *five* problems but *no more than two problems from any one section.* Give complete proofs. A correct solution will **not** consist in merely quoting a theorem. If you attempt more than five problems, indicate clearly which five problems you would like graded. You have three hours.

$\mathbb{C}$ = the field of complex numbers
$\mathbb{Q}$ = the field of rational numbers.

1. Groups

(1) Let G be a finite group and let P be a Sylow p-subgroup of G, where p is a prime number.
   (a) If H is a normal subgroup of G, show that $P \cap H$ is a Sylow p-subgroup of H.
   (b) Give an example of a group G, a Sylow p-subgroup P of G and a subgroup H of G such that $P \cap H$ is *not* a Sylow p-subgroup of H.

(2) Classify up to isomorphism all groups of order 70. (Be as complete in your analysis as possible. In particular, be sure to prove that no two of the groups you find are isomorphic to each other.)

(3) Prove that every group of order 150 is solvable.

2. Fields

(4) Let p be an odd prime and let $E = \mathbb{Q}(\zeta + \zeta^{-1})$, where $\mathbb{Q}$ is the field of rational numbers and $\zeta \in \mathbb{C}$ is a primitive pth root of unity. Determine the Galois group $Gal(E/\mathbb{Q})$.

(5) Let E and F be subfields of $\mathbb{C}$ such that $E/\mathbb{Q}$ and $F/\mathbb{Q}$ are both finite and Galois. Let EF denote the smallest subfield of $\mathbb{C}$ containing both E and F and assume that $E \cap F = \mathbb{Q}$. Prove that

$$Gal(EF/\mathbb{Q}) \cong Gal(E/\mathbb{Q}) \times Gal(F/\mathbb{Q}).$$

(6) Let E/F be a normal field extension and $f(x) \in F[x]$ be an irreducible polynomial. Suppose $g(x)$ and $h(x)$ are monic irreducible factors of $f(x)$ in $E[x]$. Prove that there exists an automorphism σ of E/F such that $g = h^\sigma$ (where h^σ denotes the polynomial obtained by applying σ to the coefficients of h).

(7) Supply an example *and* a proof of your example for each of the following:
   (a) A unique factorization domain that contains a prime ideal that is neither a maximal ideal nor a principal ideal
   (b) Modules A, B and C over a ring R such that A is a submodule of B but $A \otimes_R C$ is not isomorphic to a subgroup of $B \otimes_R C$
   (c) Right R-modules A and B such that $\mathrm{Hom}_R(A, B) = 0$ and $\mathrm{Hom}_R(B, A) \neq 0$

(8) Let R be a ring with identity and let M be a unital right R-module. Let I and J be ideals of R. Prove
   (a) $M \otimes_R R/I \cong M/MI$ as right R/I-modules.
   (b) $R/I \otimes_R R/J \cong R/(I + J)$ as R/I-R/J bimodules.

(9) Let R be a ring and let M be a right R-module. Suppose

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors M_i/M_{i-1} are non-zero simple modules that are pairwise non-isomorphic. If X and Y are isomorphic submodules of M, prove that $X = Y$. (*Suggestion: First do the case that X is simple.*)