Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
Spring, 2008

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This exam has two parts, ordinary differential equations and partial differential equations. In each part choose four problems. Mark clearly the problems you choose and show the details of your work. Books or notes are not allowed.

Part I: Ordinary Differential Equations

1. Consider the function sequence given by $y_0(t) = 0$ and

$$y_{k+1}(t) = 1 - \int_0^t y_k(s)ds, \quad k = 0, 1, \dots$$

(a) Prove that the seqeunce converges unniformly on any closed interval $[-N, N]$, $N > 0$.

(b) Find the limit function of the sequence.

2. Let $p(x), q(x)$ be continuous functions on [a,b]. Consider the boundary value problem

$$y'' + p(x)y' + q(x)y = 0; \quad y(a) = y(b) = 0.$$

Prove that if $q(x) < 0$ on [a,b] then $y(x) = 0$ is the only solution.

3. Consider the nonlinear differential equation $\ddot{x} = -(4 + x)^{-2}$.

(a) Sketch the trajectories in the phase plane on the part $x > 0$.

(b) With the initial conditions $x(0) = 0$, $\dot{x}(0) = v_0$, find the smallest speed v_0 such that the solution exists on all $t > 0$ and $x(t) \to \infty$ as t approaches infinity.

4. Find all the eigenvalues and the corresponding eigenfunctions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0.$$

5. Solve the initial value problem.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 2e^{t} \\ -5e^{-t} \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

6. Suppose a function $\sigma(t)$ satisfies $\sigma(0) = 2$ and

$$\sigma'(t) \geq -2\sigma(t).$$

Prove that $\sigma(t) \geq 2e^{-2t}$ for all $t > 0$.

Part II: Partial Differential Equations

1. (Poisson's Formula on a Disk) Let $f(\theta)$ be a continuous and 2π-periodic function with Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta).$$

Let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta).$$

(a) Prove that the series for $u(r,\theta)$ converges uniformly on any disk $B_R = \{(r,\theta) \mid 0 \le r \le R\}$ with $R < 1$.

(b) Show how to rewrite the series for $u(r,\theta)$ in the form

$$u(r,\theta) = \int_0^{2\pi} f(\phi) P(r, \theta - \phi) d\phi$$

where P is the Poisson kernel satisfying

$$P(r,\phi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\phi + r^2}.$$

(c) Prove that $\lim_{r\to 1^-} u(r,\theta) = f(\theta)$ uniformly.

2. Solve the following initial value problem for Burger's equation.

$$\begin{aligned} u_t(x,t) + u(x,t)u_x(x,t) &= 0 \quad \text{on} \quad t > 0, \\ u(x,0) &= x^3 + 2x. \end{aligned}$$

3. Find a solution to the initial value problem for the heat equation.

$$\begin{aligned} u_t(x,t) &= 9u_{xx}(x,t) \quad \text{on} \quad t > 0, \\ u(x,0) &= 3e^{-2x^2}. \end{aligned}$$

4. Let $u(x)$ be an entire harmonic function on R^n. Prove that $u(x)$ is identically zero if $\int_{R^n} |u(x)| dx < \infty$.

5. Let $u(x)$ be a smooth harmonic function on the punctured ball in R^n

$$B = \{\ x\ \in\ R^n\ \mid\ 0 < |x| < 2\ \}.$$

Suppose that there is a positive number M such that $|u(x)| \leq M$ for all the points in B. Prove that the origin is a removable singular point for $u(x)$.

6. Solve the initial value problem

$$\begin{aligned}
u_{tt}(x,t) &= 4u_{xx}(x,t) \quad \text{on} \quad 0 < x < \infty,\ \ t > 0 \\
u(x,0) &= 2\sin(x) - \frac{1}{2}\sin(2x) \quad \text{for}\ \ 0 < x < \infty. \\
u_t(x,0) &= \sin(2x) \quad \text{for}\ \ 0 < x < \infty. \\
u(0,t) &= 0 \quad \text{for}\ \ 0 \leq t
\end{aligned}$$