Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

April 19, 2008

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let $X_1, X_2, \cdots, X_n$ be a sequence of iid random variables with $E|X_1| = \infty$. Show that

$P(limsup_{n\to\infty} n^{-1}|X_n| = \infty) = 1$.

Hint: You may use the fact that for any nonnegative random variable Y,

$EY \leq \sum_{n=1}^{\infty} P(Y \geq n)$.

2. Suppose that $X \sim N(\theta, 1)$ where $\theta \in \Theta = \mathcal{R}$. Consider first a decision problem with $\mathcal{A}$=Θ and $L(\theta, a) = (1 - \theta a)^2$.

1). Find a generalized Bayes rule in this problem against Lebesgue measure on Θ.

2). Argue that every finite nonrandomized decision rule $\delta(x)$ is Bayes versus Δ a point mass prior concentrated at 0.

The following fact can be used to finish part 3) without proving it:

Under the the loss loss function, L in part 1), the rule

$\delta(x) = \frac{x}{1+x^2(\frac{\tau^2}{1+\tau^2})}$ is Bayes vs a $N(0, \tau^2)$ prior.

3). Consider now the squared error loss estimation of $\frac{1}{\theta}$, i.e., a second decision problem with $\Theta = \mathcal{R}\,/\{0\}$, $\mathcal{A}$=Θ and loss $L^*(\theta, a) = (\frac{1}{\theta} - a)^2$. show that for $c \in (0, 1)$, the estimator $\delta_c(x) = \frac{x}{1+cx^2}$ is admissible for this second problem.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A, B \in \mathcal{F}$.

(a) Show that $P(A \cap B) \geq P(A) + P(B) - 1$.

(b) Show that $|P(A) - P(B)| \leq P(A \Delta B)$.

4. Let $X_1, \ldots, X_n$ be a random sample from a population distribution on $\mathbf{R}$ with desnity function given by

$$f(x, \theta) = \frac{1}{\theta} e^{-\frac{x-\theta}{\theta}} I_{(\theta, \infty)}(x),$$

where $\theta > 0$ is an unknown parameter.

(a) Find a sufficient statistic for θ.

(b) Find a minimal sufficient statistic for θ.

(c) Determine if the minimal sufficient statistic found in part (b) is a complete statistic for θ. Explain your reasoning.