Ph.D. Qualifying Exam: Real Analysis

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Instructions: Do six of the 9 questions. No materials are allowed.
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1. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a finite measure space and that f_n is a sequence in $L^1(\Omega, \mathcal{F}, \mu)$ which converges to 0 in $L^1(\Omega, \mathcal{F}, \mu)$.
   (a) Give an example to show that f_n need not converge to 0 almost everywhere.
   (b) Show that f_n converges in measure to 0.
   (c) Suppose that some subsequence of the f_n converges pointwise almost everywhere to some function f. Must $f = 0$ almost everywhere? Explain.

2. Suppose f and g are nonnegative integrable functions defined on a measure space $(\Omega, \mathcal{F}, \mu)$.
   (a) Show that $\min\{\int_\Omega f\,d\mu, \int_\Omega g\,d\mu\} \geq \int_\Omega \min\{f, g\}\,d\mu$.
   (b) If equality holds then what can be said about the relationship between f and g?

3. (a) Give an example of a sequence of bounded functions which are Riemann integrable on a compact interval $[a, b]$ and the sequence converges pointwise to a function which is not Riemann integrable.
   (b) Give an example of a function f which is not Lebesgue measurable on $[a, b]$ but f^2 is.
   (c) Give an example of a function f which is Lebesgue integrable on $[a, b]$ but f^2 is not.

4. (a) State the Baire Category theorem. If you use the terminology "first category" or "second category" then you should define those terms.

(b) Suppose that E is a complete metric space with metric d (CORRECTION: Suppose E is a Banach space). Suppose that $X \subseteq E$ has the property that it complement X^c is countable. Show that X is set of the second category.

5. Prove or disprove.

   (a) Every absolutely continuous function defined on [0,1] is of bounded variation.

   (b) Every continuous function defined on [0,1] is of bounded variation.

   (c) If f is continuous and increasing on [0,1] then $f(1) - f(0) = \int_0^1 f'(x)\, dx$.

6. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space and f_n is a sequence of real valued Borel measurable functions $f_n : \Omega \to \mathbb{R}$ such that

$$\sum_{n \in \mathbb{N}} \int_\Omega |f_n|\, d\mu < \infty$$

   Show that $\sum_n f_n(x)$ converges μ-almost everywhere to a function $f(x)$ say and $f \in L^1(\mu)$ and

$$\int_\Omega f\, d\mu = \sum_{n \in \mathbb{N}} \int_\Omega f_n\, d\mu$$

7. Consider the sequence $f_n(x) = e^{-n\sqrt{x}}$. Show that, for any $a > 0$ f_n converges to 0 uniformly on $[a, \infty)$ but f_n does not converge uniformly on $(0, \infty)$. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx$$

   and explain your answer.

8. Let $I = [0, 1]$ and $K = I^n$. Fix α such that $0 < \alpha < 1$. Let S be the family of all real valued functions on K for which

$$\|f\|_\alpha = \left(\sup_K |f(x)| + \sup_{K\times K} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right) \leq 1.$$

   Show that the closure of S in $C(K)$, the space of continuous functions on K with the supremum norm, is compact.

9. Let f be a continuous function on $[0, \infty)$ and $\int_0^\infty |f(x)| dx < \infty$.. Assume that $\int_0^\infty f(x)e^{-nx} dx = 0$ for all integers n sufficiently large. Show that $f \equiv 0$.