Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
Spring II, 2009

This exam has two parts, ordinary differential equations and partial differential equations. In Part I, do problems 1 and 2 and choose two from the remaining problems. In Part II, Choose four problems.

Part I: Ordinary Differential Equations

1. Consider the differential equation with initial condition

$$dx/dt = F(t, x), \quad x(a) = x_0 \in R^n$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and
$F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_n(t, x))^T$. Suppose $F(t, x)$ is continuous for $a \le t \le b$ and $x \in R^n$ and satisfies a Lipschitz condition $|F(t, x) - F(t, y)| \le L|x - y|$ for $a \le t \le b$ and all x, y.

(a) Convert the differential equation with the initial condition into an equivalent integral equation.

(b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval [a, b] to a limit function $x_\infty(t)$.

(c) Show that $x_\infty(t)$ is a solution to the differential equation on [a, b].

(d) Establish that the solution to the differential equation with the given initial condition is unique.

2. Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval where $p(x), q(x)$ are continuous.

1) Let $y(x)$ be a non-zero solution. Prove that all the roots of $y(x)$ are isolated.

2) Let $y_1(x), y_2(x)$ be two linearly independent solutions. Prove that there is exactly one root of $y_2(x)$ between any two roots of $y_1(x)$.

3. Prove that any non-trivial solution to the Airy's equation $y'' + xy = 0$ has infinitely many zeros on $(0, \infty)$ but at most one zero on $(-\infty, 0)$. Find the power series solution satisfying the following initial conditions $y(0) = 1, y'(0) = -1$.

4. Consider the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y'(0) = 0, y(\pi) = y'(\pi).$$

(a) Find all eigenfunctions and eigenvalues $(y_n(x), \lambda_n)$. Be sure to check the possibilities $\lambda_n \leq 0$.

(b) Show that the set of eigenfunctions form an orthogonal set of functions in $L^2[0, \pi]$.

(c) Solve for the λ_n graphically and make a good asymptotic estimate for λ_n.

5. (a) Find the fundamental solution matrix for the linear system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

(b) Solve the initial value problem.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

6. Consider the nonlinear DE $\ddot{x} + 4\sin(x) = 0$.

(a) Find the integral curves and sketch the trajectories in the phase plane.

(b) Classify the critical points as vortex points or saddle points of the autonomous system defining these curves.

Part II: Partial Differential Equations

1. Consider the Fourier series solution to the heat equation with an initial condition and a boundary condition

$$\begin{cases} u_t = u_{xx}, & \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = f(x), \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

(a) Derive the formal solution $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$ where the b_k are the coefficients of the Fourier series $\sum_{k=1}^{\infty} b_k \sin kx$ for the continuous function $f(x)$.

(b) Show that for every $\delta > 0$ this solution series converges uniformly in the region $0 \leq x \leq \pi, t \geq \delta$. Also show the same convergence for the corresponding series for u_t, u_x, u_{xx}.

(c) Show that $\lim_{t \to 0^+} u(x,t) = f(x)$ in the L^2 norm.

2. Find all radially symmetric solutions of $\Delta u + u = 0$ in R^3.

3. Consider the eikonal equation $u_x^2 + u_y^2 = u^2$.

(a) Find all solutions of the form $u(x,y) = f(x)$.

(b) Use (a) to write down a general solution $u = u(x,y,a,b)$. (Hint: Use the fact that the PDE is invariant under rotations in the xy)

(c) Find the solution of the PDE satisfying the condition $u(x,x) = 2$.

4. Let $B^+ = \{(x,y) \mid x^2 + y^2 < 1, y > 0\}$ be the open half disk. Suppose $u(x,y) \in C^2(B^+) \cap C^1(\bar{B^+})$ satisfies $\Delta u = u_{xx} + u_{yy} = 0$ in B^+ and $u_y(x,0) = 0$. Prove that the extension $u(x,y)$ to B defined as follows is a harmonic function on all B.

$$u(x,y) = \begin{cases} u(x,y) & \text{if } y \geq 0; \\ u(x,-y) & \text{if } y < 0. \end{cases}$$

5. (Removable Singularity for Harmonic Functions) Suppose $u(x,y)$ is a smooth harmonic function satisfying $|u(x,y)| \leq M$ for some constant M on the deleted disk $0 < \sqrt{x^2+y^2} < 2$. Prove that the singularity at the origin is removable.

6. Solve the following initial value problem for Burger's equation.

$$\begin{aligned} u_t(x,t) + u(x,t)u_x(x,t) &= 0 \text{ on } t > 0, \\ u(x,0) &= 2x - 1. \end{aligned}$$