Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

April 11, 2009

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let F be a cumulative distribution function on the real line $\mathbf{R}$ and $a \in \mathbf{R}$. Show that

$$\int [F(x+a) - F(x)]dx = a.$$

2. Let $X_1, \ldots, X_n$ be independent random vectors, and let $\mathcal{U}$ be the set of all variables of the form

$$\sum_{i=1}^{n} g_i(X_i),$$

for arbitrary measurable functions g_i with $E\{g_i^2(X_i)\} < \infty$. Show that the projection of an arbitrary random variable T with finite second moment onto the class $\mathcal{U}$ is given by

$$S = \sum_{i=1}^{n} E(T|X_i) - (n-1)E(T).$$

3.

Let $X_1, \cdots, X_n$ be iid random variables with common probability mass function (pmf)

$f(x; \theta) = \theta^{x-1}(1+\theta^2)^{-\frac{x+1}{2}}, x = 1, 3, 5, \cdots, \theta > 0.$

1). Write down the joint pmf of $X_1, \cdots, X_n$ and show that it belongs to an Exponential family of distributions.

2). Show that $T = \sum X_i$ is sufficient for θ.

3). Show that T is complete for θ.

(Note: you need to verify the relevant condition(s) guaranteeing the completeness of T.)

4). Find the distribution of $Y_1 = \frac{X_1 - 1}{2}$.

5). Find the uniformly minimum variance unbiased estimators (UMVUEs) of (a) $\gamma_1(\theta) = \theta^2$ and (b) $\gamma_2(\theta) = \theta^4$.