Topology Ph.D. Qualifying Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. Let $f : A \mapsto Y$ be a map of topological spaces with f being continuous and Y Hausdorff. Suppose that $A \subset X$ where X is a topological space. Prove that f has at most one continuous extension to a map from the closure $\overline{A}$ to Y.

2. Let $f : X \mapsto Y$ be a map of sets. Define the graph of f. Now let $f : X \mapsto Y$ be a map of topological spaces with f being continuous and Y Hausdorff. Prove that the graph of f is closed.

3. Define what it means for (X, d) to be a metric space. Then $d : X \times X :\to \mathbb{R}$: is d continuous? Discuss.

4. Define the term *identification map* in the category of topological spaces. Let $\pi : X \to Y$ be a surjective, continuous map of topological spaces. Suppose that π maps closed sets to closed sets. Show that π is an identification map. What happens if we replace closed sets by open sets? Justify your answers.

5. Prove that a topological space X is Hausdorff if and only if the diagonal $:= \{(x, x)$ such that $x \in X\}$ is closed in the product $X \times X$.

6. Prove that the closed interval $[0, 1]$ considered as a subset of $\mathbb{R}$ in the usual topology is compact.

7. Prove that $\mathbb{R}$ with the usual topology is connected.

8. Let G be a topological group that acts on a topological space X. Prove that if both G and the quotient space are X/G connected then X is connected.

9. A topological space X is said to be *locally connected* if the connected components of each point form a base of neighborhoods of X. Prove that in a locally connected space the connected components of X are both closed and open in X.

10. Let A be a subspace of a topological space X. Prove that A is disconnected if and only if there exist two closed subsets F and G of X such that $A \subset F \cup G$ and $F \cap G \subset X - A$.

11. A topological space X is said to be *regular* if a singleton set and closed set that are disjoint can be separated by disjoint open sets. Prove that in a regular space disjoint closed and compact sets can be separated by disjoint open sets.

12. Let $X = \Pi_{\mu \in M} X_\mu$ and $Y = \Pi_{\mu \in M} Y_\mu$ be the Cartesian products of the topological spaces $(X_\mu)_{\mu \in M}$ and $(Y_\mu)_{\mu \in M}$ and let X and Y have the product topologies, respectively. Prove that if for each $\mu \in M$ the maps $f_\mu : X_\mu \to Y_\mu$ are continuous then $f : X \to Y$ defined by $f(x)_\mu = f_\mu(x_\mu)$ is continuous.

2 Part Two: Do three questions

1. (i) Define what is meant by a Möbius band. Identify the space obtained by identifying the boundary of a Möbius band to a point. Give a brief explanation.

   (ii) Let X be a topological space and let $x_0 \in X$. Define the product of homotopy classes of loops $[\alpha]_{x_0}$ based at x_0 and verify in detail that this product is associative.

2. Give the definitions of deformation retract and strong deformation retract for topological spaces. Compute the fundamental group of $\mathbb{R}^3 - C$ where C denotes the circle $x^2 + y^2 = 1, z = 0$.

3. (i) The polygonal symbol of a certain surface without boundary is $zy^{-1}xyzx^{-1}$. Identify the surface. What is its Euler characteristic?

   (ii) Explain how polygons with an even number of sides may be used to classify surfaces without boundary. You do not need to give detailed proofs.

4. It is known that if $p : \tilde{X} :\to X$ is a covering space and $x_0 \in X$ then the cardinality of $p^{-1}(x_0)$ is the index of $p_* \pi_1(\tilde{X}, y_0)$ in $\pi_1(X, x_0)$ where $p(y_0) = x_0$. Use this fact to deduce that there is no n-sheeted covering of the circle S^1 for any finite n.

5. Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ be the closed unit disk. Prove that D cannot be retracted to the unit circle S^1. Deduce that any continuous map $f : D \longrightarrow D$ has a fixed point. (*Hint:* Consider the line joining x to $f(x)$ where $x \in D$.)

6. Let P^2 be the two-dimensional real projective space and T^2 be the two-dimensional torus.

   (i) What is $\pi_1(P^2)$? Explain your answer.

   (ii) The space P^2 can be obtained from the unit disk D^2 by identifying $x \sim -x$ if $||x|| = 1$. Let $p \in \text{int}(D^2)$, the interior of D^2. Find the fundamental group of $P^2 - \{[p]\}$.

   (iii) Let $f : P^2 \mapsto T^2$ be a continuous map. Show that f is null homotopic, that is, is homotopic to a constant map.

7. Let S^1 denote the unit circle in R^2. Define $f : S^1 \mapsto S^1$ by $f(x) = -x$. Prove that f is homotopic to the identity map. Now suppose that $g : S^1 \mapsto S^1$ is a map that is not homotopic to the identity map. Show that $g(x) = -x$ for some $x \in S^1$.