Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

April 24, 2010

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let f_n and g_n be integrable functions for a measure μ with $|f_n| \leq g_n$. Suppose that as $n \to \infty$, $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for almost all x. If $\int g_n d\mu \to \int g d\mu$, then $\int f_n d\mu \to \int f d\mu$.

2. Let $\mu = E(Y)$ denote the mean of a response variable Y. When the response variable Y is subject to missing, we do not observe all $Y_1, \ldots, Y_n$ in the sample. Let D represent the missing indicator variable which is equal to 1 if Y is observed and is equal to 0 if Y is missing, and let $(Y_1, D_1), \ldots, (Y_n, D_n)$ denote a random sample of (Y, D). The complete-case sample mean of the observed Y-values is defined by

$$\hat{\mu}_c = \frac{\sum_{i=1}^n D_i Y_i}{\sum_{i=1}^n D_i}.$$

Let $\pi = E(D) = P(D = 1)$ denote the probability of observing Y so that $1 - \pi$ represents the missing proportion of Y. We assume $\pi > 0$ so that there is positive probability of observing Y.

(a) Show that $\hat{\mu}_c$ is a consistent estimator of $\mu_1 = E(Y|D = 1)$.

(b) Find the asymptotic variance σ_c^2 of $\hat{\mu}_c$.

(c) Find a consistent estimator of σ_c^2 in part (b).

(d) Show that μ_1 is greater than μ if $\pi(y) = P(D = 1|Y = y)$ is a strictly increasing function in y.

(e) Show that when missing is completely at random, $\hat{\mu}_c$ is a consistent estimator of μ. In this case, find the asymptotic variance of $\hat{\mu}_c$ and compare it with the variance of the full-data sample mean $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i$.

3. (a) For any two events $A, B \in \mathcal{F}$ prove that

$$2\mathrm{P}(AB) \leq \mathrm{P}(A) + \mathrm{P}(B).$$

(b) For integrable X prove that

$$\mathrm{E}(X) = \int_0^\infty \mathrm{P}(X > t)dt - \int_{-\infty}^0 \mathrm{P}(X < t)dt.$$