Topology Ph.D. Qualifying Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. Prove that a compact subspace of a Hausdorff space is closed.
2. Prove that in a complete metric space (X, d) a subspace Y of X is complete if and only if it is a closed subspace of X.
3. Let Y be a Hausdorff space; and $f, g : X \mapsto Y$ be continuous maps. Suppose $f = g$ on a subset A of X which is dense in X. Prove that $f = g$ on X.
4. Let X, Y be topological spaces. Suppose that X is compact and Y is Hausdorff. Let $f : X \mapsto Y$ be continuous and bijective. Prove that f is a homeomorphism.
5. Prove that a space Y is Hausdorff if and only if for every topological space X and every pair of continuous functions $f : X \mapsto Y$ and $g : X \mapsto Y$, the set $\{x \in X | f(x) = g(x)\}$ is closed in X.
6. Prove that a closed map is a quotient map.
7. For each $\alpha \in A$, let X_α be a topological space with topology $\mathfrak{T}_\alpha$. Let $X = \Pi_{\alpha \in A} X_\alpha$ be given the product topology, and let $\pi_\alpha : X \mapsto X_\alpha$ denote the projection map. Let Y be a topological space. Prove that a function $f : Y \mapsto X$ is continuous if each $\pi_\alpha \circ f$ is continuous.
8. Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a sequence of points in a topological space X, converging to x_0. Prove that the set $K = \{x_n : n = 0, 1, 2, 3, \cdots\}$ is compact.
9. Prove that if two connected sets A and B in a space X have a common point p, then $A \cup B$ is connected.
10. Let X be a topological space. Let $A \subset X$ be connected. Prove that the closure $\overline{A}$ of A is connected.

11. Prove that a path-connected topological space is connected.

12. Prove or Disprove: A quotient space of a Hausdorff space is Hausdorff.

2 Part Two: Do three questions

1. Let $\mathbb{D}^n$ be the n-dimensional ball i.e. $\mathbb{D}^n = \{\mathbf{x} \in \mathbb{R}^n$ such that $\sqrt{x_1^2 + x_2^2 + ... + x_n^2} \leq 1$ and $\mathbb{S}^{n-1}$ the $(n-1)$-sphere, realized as the boundary of $\mathbb{D}^n$ i.e. $\{\mathbf{x} \in \mathbb{R}^n$ such that $\sqrt{x_1^2 + x_2^2 + ... + x_n^2} = 1\}$. Prove that the following are equivalent:

   1. There is no retraction $\mathbb{D}^n \mapsto \mathbb{S}^{n-1}$.

   2. Every continuous map $\mathbb{D}^n \mapsto \mathbb{D}^n$ has a fixed point.

2. Show that there is no one to one continuous map $\mathbb{R}^n \mapsto \mathbb{R}^2$ for $n > 2$ with $f(0) = 0$. (Hint: Consider the induced map $f : \mathbb{R}^n \backslash \{0\} \mapsto \mathbb{R}^2 \backslash \{0\}$ and $f_* : \pi_1(\mathbb{R}^n \backslash \{0\}) \mapsto \pi_1(\mathbb{R}^2 \backslash \{0\})$.)

3. Let $\Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ be the diagonal subspace. Show that the projection from the complement of Δ

$$\mathbb{S}^2 \times \mathbb{S}^2 - \Delta \mapsto \mathbb{S}^2$$

   given by $(x, y) \mapsto x$ is a homotopy equivalence.

4. Let $X = \mathbb{S}^2 \cup J$ where $J = \{(0, 0, z) \in \mathbb{R}^3 : -1 \leq z \leq 1\}$ is the interval on the z-axis joining the north and south poles. Compute the fundamental group of X.

5. The polygonal symbol of a certain surface without boundary is $acb^{-1}a^{-1}cb$. Identify the surface. What is its Euler characteristic?

6. Given a path f in a space X from x_0 to x_1, let $\widetilde{f}$ be the path in X defined by $\widetilde{f}(s) = f(1-s)$. Prove that $f \cdot \widetilde{f}$ ($\cdot$ is the path multiplication) is homotopic to the constant path c_{x_0} at x_0.

7. Let X be the space where one point is removed from the surface of a torus. Compute the fundamental group of X.

8. Give the definition of a covering map $p : \widetilde{X} \mapsto X$. Prove that the cardinality of the fibers $p^{-1}(q)$ is the same for all fibers.