Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Consider the differential equation of the first order with initial condition

$$\frac{dx}{dt} = F(t,x), \quad x(a) = x_0 \in R^n$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ and
$F(t,x) = (F_1(t,x), F_2(t,x), \dots, F_n(t,x))^T$. Suppose $F(t,x)$ is continuous for $a \leq t \leq b$ and $x \in R^n$ and satisfies a Lipschitz condition $\|F(t,x) - F(t,y)\| \leq L\|x-y\|$ for $a \leq t \leq b$ and all x, y, where $\|\cdot\|$ denotes the standard norm in R^n.

(a) Convert the differential equation with the initial condition into an equivalent integral equation.

(b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval [a, b] to a limit function $x_\infty(t)$.

(c) Show that $x_\infty(t)$ is a solution to the differential equation on [a, b].

(d) Establish that the solution to the differential equation with the given initial condition is unique.

2. A gradient system in R^n is a system of ODEs of the form

$$\frac{dx(t)}{dt} = -\nabla V(x(t))$$

for some smooth function $V(x)$ $(x \in R^n)$ and ∇ is the gradient operator. Recall that a periodic orbit for such a system is a *non-constant* solution $x(t)$, such that $x(t) = x(t+T)$ for some $T > 0$. Show that a gradient system can not have periodic orbits.

3. Find the power series solution of $y'' - xy = 0$ satisfying the initial conditions $y(0) = 2, y'(0) = 0$. Determine the radius of convergence for the power series solution.

4. Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem (the eigenvalues are expressed in terms of the roots of a transcendental equation that can not be solved exactly, so estimate graphically the position of these roots):

$$y''(x) + \lambda y(x) = 0,\ 0 < x < 1,$$

$$y'(0) - y(0) = 0,\ y(1) = 0.$$

5. Solve the nonhomogeneous linear system for $x \in R^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6. Consider the nonlinear DE $\ddot{x} + 9\sin(x) = 0$.

(a) Find the integral curves and sketch the trajectories in the phase plane.

(b) Show that the solution satisfying the initial conditions $x(0) = \pi/6, \dot{x}(0) = 0$ is a periodic function of t and find an expression for the period.

7. Find Green's function $G(t, \tau)$ for the differential equation $u'' - 4u = 0$. (Recall that $G(t, \tau)$ satisfies 1) $G(t, \tau) = 0$ if $t < \tau$; 2) $G_{tt} + G = 0$ if $t > \tau$; and 3) $G(\tau, \tau) = 0, G_t(\tau, \tau) = 1$.)

Part II: Partial Differential Equations

1. Consider the Fourier series solution to the heat equation with an initial condition and a boundary condition

$$\begin{cases} u_t = u_{xx}, \quad \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = f(x), \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

(a) Derive the formal solution $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$ where the b_k are the coefficients of the Fourier series $\sum_{k=1}^{\infty} b_k \sin kx$ for the continuous function $f(x)$.

(b) Show that for every $\delta > 0$ this solution series converges uniformly in the region $0 \leq x \leq \pi, t \geq \delta$. Also show the same convergence for the corresponding series for u_t, u_x, u_{xx}.

(c) Show that $\lim_{t \to 0^+} u(x,t) = f(x)$ in the L^2 norm.

2. Find all radially symmetric solutions of $\Delta u = 0$ in $R^n (n \geq 2)$.

3. Find the equation of the surface $z = z(x,y)$ which satisfies the first order PDE

$$4yz(x,y)\frac{\partial z(x,y)}{\partial x} + \frac{\partial z(x,y)}{\partial y} + 2y = 0,$$

and contains the ellipse $y^2 + z^2 = 1, x + z = 2$.

4. Let $B^+ = \{(x,y) \mid x^2 + y^2 < 1, y > 0\}$ be the open half disk. Suppose $u(x,y) \in C^2(B^+) \cap C^0(\bar{B^+})$ satisfies $\Delta u = u_{xx} + u_{yy} = 0$ in B^+ and $u(x,0) = 0$. Prove that the extension $u(x,y)$ to B defined as follows is a harmonic function on all B.

$$u(x,y) = \begin{cases} u(x,y) & \text{if } y \geq 0; \\ -u(x,-y) & \text{if } y < 0. \end{cases}$$

5. (Removable Singularity for Harmonic Functions) Suppose $u(x)$ is a smooth harmonic function on a punctured neighborhood of the origin with the origin being a possible singularity. Prove that if $|u(x)| \leq M$ for some constant M on the punctured neighborhood, then the singularity at the origin is removable.

6. Solve the following problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (x > 0,\ t > 0),$$

$$u(0,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u(x,t)}{\partial t} = g(x),\ (x > 0).$$

7. Show that the following partial differential equation is elliptic inside the unit ball $B_1 = \{\, x \,|\, |x| < 1 \,\}$ and is hyperbolic outside the unit ball.

$$\sum_{i,j=1}^{n} (\delta_{i,j} - x_i x_j) \frac{\partial^2}{\partial x_i \partial x_j} u = 0.$$