Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

January 17, 2015

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. (20 points) Suppose the ith light bulb burns for an amount of time X_i and then remains burned out for time Y_i before being replaced. Suppose the X_i, Y_i are positive and independent with the X's having distribution F and the Y's having distribution G, both of which have finite mean. Let R_t be the amount of time in $[0, t]$ that we have a working light bulb. Show that $R_t/t \to EX_i/(EX_i + EY_i)$ almost surely.

2. (30 points) Let $X = (X_1, ..., X_n) \sim_{iid} E(a, \theta)$ with $a \in \mathbf{R}$ and $\theta > 0$.
a. (10 points) Find the UMVUE of a when θ is known.
b. (10 points) Find the UMVUE of θ when a is known.
c. (10 points) Assume that θ is known. Find the UMVUE of $P[X_1 \geq t]$ and $\frac{d}{dt} P[X_1 \geq t]$ for a fixed $t > 0$.

3. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables. Prove that if $X_1 \leq X_2 \leq \cdots$ and $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{\text{a.s.}} X$.

4. Let $X_1, \ldots, X_n$ be independent and identically distributed according to the normal $N(\theta, 1)$ distribution, where $\theta \in \Theta = (-\infty, \infty)$.

(a) Find a minimal sufficient statistic for θ.

(b) Suppose the interest is in estimation of θ^2. Calculate the Fisher information $I(\theta^2)$ contained in $X_1, \ldots, X_n$ about θ^2.

(c) Find the UMVU estimator of θ^4 and its variance. Does the UMVU estimator achieve the Cramér-Rao lower bound? Explain your reasoning.

(d) Suppose we want to test the null hypothesis $H_0 : \theta = \theta_0$ versus the alternative hypothesis $H_a : \theta \neq \theta_0$, where θ_0 is a fixed number. Find the likelihood ratio test of size α.

(e) Let $a_1, \ldots, a_n$ be non-random real numbers. Show that a necessary and sufficient condition for $\sum_{i=1}^n a_i X_i$ and $\sum_{i=1}^n X_i$ to be independent is $\sum_{i=1}^n a_i = 0$.