Real Analysis, Ph.D. Qualifying Exam

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January 09, 2016

Instructions: Do six of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours.

1. Let f be in $L^1(\mathbb{R})$. Let g be the function defined by

$$g(x) = \int_0^1 t^2 f(x+t)\, dt$$

for $x \in \mathbb{R}$. Show that g is continuous on $\mathbb{R}$.

2. Let $1 < p < \infty$. Suppose that f is a measurable function on $[0, \infty)$ with the property that $\int_0^\infty |f(x)|^p\, dx < \infty$. Show that

$$\lim_{x\to\infty} x^{\frac{1}{p}} \int_x^\infty \frac{f(t)}{t}\, dt = 0.$$

3. Suppose that a sequence $\{f_n\} \subset L^1([0,1])$ satisfies $\|f_n\|_1 \le 1$ for all $n \ge 1$. (Here $\|\cdot\|_1$ denotes the norm in $L^1([0,1])$.) Show that $\frac{f_n(x)}{n^2} \to 0$ as $n \to \infty$ for almost every $x \in [0,1]$.

4. Prove or give a counterexample in Parts (a) and (b).

(a) There exists a continuous function $f : [0,1] \longrightarrow \mathbb{R}$ so that

$$\int_0^1 x f(x)\, dx = 1 \quad \text{and} \quad \int_0^1 x^{2n} f(x)\, dx = 0, \quad \text{for all } n = 1, 2, 3 \ldots.$$

(b) There exists a continuous function $f : [-1,1] \longrightarrow \mathbb{R}$ so that

$$\int_{-1}^1 x f(x)\, dx = 1 \quad \text{and} \quad \int_{-1}^1 x^{2n} f(x)\, dx = 0, \quad \text{for all } n = 1, 2, 3 \ldots.$$

5. Let S be the set of all continuously differentiable functions $f : [0,1] \longrightarrow \mathbb{R}$ such that

$$|f(x)| + |f'(x)| \le 1 \text{ for all } 0 \le x \le 1.$$

Show that S has a compact closure in $C([0,1])$ with norm $\|\cdot\|_\infty$, where

$$\|g\|_\infty = \sup_{0\le x\le 1} |g(x)|.$$

6. Let f be in $L^1([0,1])$. Find the limit

$$\lim_{n\to\infty} \int_0^1 |f(x)|^{1/n}\, dx.$$

Justify your answer.

7. Let $f : [0,1] \to \mathbb{R}$ be a function such that there exists a constant $M \geq 0$ for which

$$|f(x) - f(y)| \leq M|x-y| \text{ for all } x, y \in [0,1].$$

Show that f maps sets of Lebesgue measure zero to sets of Lebesgue measure zero.

8. Show that there is no real number $M \geq 0$ for which

$$\sup_{0\leq x\leq 1} |f(x)| \leq M \int_0^1 |f(x)|\, dx$$

for all $f \in C([0,1])$.