ALGEBRA PH.D. QUALIFYING EXAM, APRIL 22, 2017

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Please do six problems, with three problem from each of the sections. Give complete proofs. If you attempt more than six problems, indicate clearly which six problems you would like to be graded. You have three hours to complete the exam.

1. Groups

1. Show that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4 contained in A_4 such that $S_4/N \cong S_3$ and $A_4/N \cong Z_3$.

2. (a) Let G be a group and let $Z(G)$ be its center. Show that if $G/Z(G)$ is cyclic, then G is abelian.

(b) Show that if G is a nonabelian finite group, then $|Z(G)| \leq \frac{1}{4}|G|$. Give an example of G with $Z(G) = \frac{1}{4}|G|$.

3. Let G be a group with a normal subgroup N of order 5, such that $G/N \cong S_3$. Show that $|G| = 30$, G has a normal subgroup of order 15, and G has 3 subgroups of order 10 that are not normal.

4. Show that a group of order $3^3 \cdot 5 \cdot 13$ must have a normal Sylow 13-subgroup or a normal Sylow 5-subgroup. [Hint: Show that if a Sylow 13-subgroup is not normal, then a Sylow 13-subgroup must normalize a Sylow 5-subgroup. Consider the normalizer of a Sylow 5-subgroup.]

2. Rings. Fields.

5. Let R be a finite commutative ring with more than one element and no zero-divisors. Show that R is a field.

6. Find all values of a in $\mathbb{Z}_5$ such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)$$

is a field. Justify your answer.

7. Let $D = \mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} \mid m, n \in \mathbb{Z}\}$ - a subring of the field of real numbers and necessarily an integral domain (you need not show this) - and $F = \mathbb{Q}(\sqrt{5})$ its field of fractions. Show the following:

(a) $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.

(b) D is not a unique factorization domain.

8. Let p be a prime and let R be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ pb & a \end{pmatrix}$ where $a, b \in \mathbb{Z}$.

(a) Prove that R is a ring

(b) Prove that R is isomorphic to $\mathbb{Z}[\sqrt{p}]$.