Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Solve the nonhomogeneous linear system for $x \in \mathbb{R}^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} e^{-t} \\ 2 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

2. Suppose $y(t)$ satisfies $y'(t) \leq 2y(t) + 1$ and $y(0) = 1$. Prove that $y(t) \leq (3e^{2t} - 1)/2$ for all $t \geq 0$.

3. Draw the phase portrait of the differential equation on the half plane $x \geq 0$. Find the smallest v_0 such that the solution $x(t)$ of the equation with the initial conditions $x(0) = 0, \dot{x}(0) = v_0$ satisfies $\lim_{t\to\infty} x(t) = \infty$.

$$\ddot{x} = -\frac{4}{(4+x)^2}$$

4. Consider the initial value problem

$$\frac{dx}{dt} = \begin{cases} -x \ln |x|, & x \neq 0 \\ 0, & x = 0 \end{cases}, \qquad x(0) = x_0.$$

where $x \in \mathbb{R}$.

(a) Explain if there is a solution with $x_0 = 0$.

(b) If a solution with $x_0 = 0$ exists, is it unique? Prove it or provide a counterexample.

5. Prove that the planar system

$$\begin{cases} \dot{x} &= x - 2y - x^3 - xy^2, \\ \dot{y} &= 2x + y - y^3 - x^2 y, \end{cases}$$

has a unique equilibrium that is unstable and a unique limit cycle that is stable.

Hint: Polar coordinates.

6. Consider the Liénard equation $\ddot{u} + \dot{u} + g(u) = 0$, where g is C^1 for $|u| < K$ with some constant $K > 0$, and $ug(u) > 0$ if $u \neq 0$. Convert it into an equivalent system by letting $x = u$ and $y = \dot{u}$, and prove that the origin $(0, 0)$ is a locally asymptotically stable equilibrium of the equivalent system.

7. Consider the linear system

$$\dot{x} = (A + B(t))x, \quad x \in \mathbb{R}^n,$$

where A is an $n \times n$ constant matrix, and $B(t)$ is a continuous, $n \times n$ matrix-valued function for all $t \in \mathbb{R}$. Suppose that: **(H1)** all eigenvalues of A have negative real part; **(H2)** $\int_0^\infty \|B(s)\| ds < \infty$. Prove that the solution $x(t) = 0$ is asymptotically stable.

Part II: Partial Differential Equations

1. Let $\phi(x) = \pi - |x|$ on $[-\pi, \pi]$ and is periodic with period 2π. Find a solution to the heat equation with the initial values.

$$\begin{cases} u_t = 4u_{xx}, & \text{for } -\infty < x < \infty,\ t > 0, \\ u(x,0) = \phi(x), & \text{for } -\infty < x < \infty. \end{cases}$$

2. Let Ω be a bounded domain in $\mathbb{R}^n$. Assume $u(x)$ is in $C^2(\Omega) \cap C(\bar{\Omega})$ and satisfies $\Delta u \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$. Show that $u \leq 0$ on $\bar{\Omega}$.

3. Solve the initial value problem for Burger's equation.

$$u_t(x,t) + uu_x(x,t) = 0,\ t > 0,$$

$$u(x,0) = 2x - 1,\ x > 0.$$

4. Solve the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & x > 0, t > 0, \\ u(x,0) = g(x), u_t(x,0) = h(x), & x > 0, \\ u_x(0,t) = 0, & t > 0, \end{cases}$$

where $a > 0$.

5. Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. Let $f \in C(\mathbb{R}^n_+)$, and $g \in C(\mathbb{R}^{n-1})$. Prove that there exists at most one bounded solution $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ of the boundary value problem

$$\Delta u = f \ \text{ in } \ \mathbb{R}^n_+; \qquad u = g \ \text{ if } \ x_n = 0.$$

6. Consider the formal solution $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$ to the heat equation with initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, t > 0, \\ u(x,0) = f(x), & 0 \leq x \leq \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \end{cases}$$

where $f \in L^2[0,\pi]$, and b_n's are the coefficients of the Fourier series $\sum_{n=1}^{\infty} b_n \sin(nx)$ for the function $f(x)$.

(a) Show that the formal solution $u \in C^\infty([0,\pi] \times (0,\infty))$, and satisfies the heat equation and boundary conditions.

(b) Show that $\lim_{t \to 0^+} u(x,t) = f(x)$ in L^2-norm.

7. Let $\Omega = \mathbb{R}^n \times (0,\infty)$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that u solves $u_{tt} - a^2 \Delta u = 0$ in Ω where $a > 0$. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and consider the cone $K = \{(x,t) : |x - x_0| \le a(t_0 - t), 0 \le t \le t_0\}$. Prove that $u \equiv 0$ within K, if $u \equiv u_t \equiv 0$ on $\{(x,t) : |x - x_0| \le at_0, t = 0\}$.