Department of Mathematics and Statistics
The University of Toledo

Ph. D. Qualifying Examination
Probability and Theory of Statistics

April 22, 2017

Instructions:

Do all two problems.
Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a three-hour closed book examination.

1. (60 pts) The probability density function of $X \sim \mathsf{Beta}(\theta, 1)$ is

$$f(x) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \theta > 0.$$

Suppose $\{X_1, \cdots, X_n\}$ is a random sample from $\mathsf{Beta}(\theta, 1)$. $\tilde{\theta}_n = \frac{\bar{X}}{1-\bar{X}}$ and $\hat{\theta}_n = -n \sum_{i=1}^n \frac{1}{\log X_i}$ are two estimators.

a. (10 pts) Find the distribution of $-\log X$, where $X \sim \mathsf{Beta}(\theta, 1)$.

b. (10 pts) Find the complete and sufficient statistic for θ.

c. (10 pts) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

d. (10 pts) Find a uniformly most powerful (UMP) level α test for

$$H_0 : \theta \leq 2 \quad \text{vs} \quad H_1 : \theta > 2.$$

Write the rejection region R of this test using the quantile of a well-known distribution.

e. (10 pts) $\hat{\theta}_n$ is the maximum likelihood estimator (MLE). Find the asymptotic distribution of $\sqrt{n}\left(\hat{\theta}_n - \theta\right)$.

f. (10 pts) Find the asymptotic distribution of $\sqrt{n}\left(\tilde{\theta}_n - \theta\right)$.

Hint: For $X \sim \mathsf{Beta}(\alpha, \beta)$, $\mathsf{E}(X) = \alpha/(\alpha + \beta)$ and $\mathsf{Var}(X) = \alpha\beta/\{(\alpha + \beta)^2(\alpha + \beta + 1)\}$.

2. (40 pts) Suppose $\{X_i, i \geq 1\}$ are independent r.v.'s such that

$$P(X_i = i) = P(X_i = -i) = \frac{1}{2i^\beta}, \beta > 0$$

and $P(X_i = 0) = 1 - \frac{1}{i^\beta}$. Define $S_n = \sum_{i=1}^n X_i$ and show that

a. (5 pts) Find $\mathsf{E}(X_i)$ and Find $\mathsf{Var}(X_i)$.

b. (15 pts) Show that if $\beta > 1$ then $S_n \to S_\infty$ a.s.

c. (20 pts) Define $s_n^2 = \sum_{i=1}^n \mathsf{Var}(X_i)$ and show that if $0 < \beta < 1$ then $\frac{S_n}{s_n} \Rightarrow N(0, 1)$. Hint: Use Lindeberg condition.