Real Analysis, Ph.D. Qualifying Exam

Željko Čučković and Trieu Le

April 15, 2017

Instructions: Do six of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours. In this exam, the measure on $\mathbb{R}$ or on any interval is Lebesgue measure.

1. Let S be the set of all continuously differentiable functions $f : [0,1] \longrightarrow \mathbb{R}$ such that $f(\frac{1}{2}) = 1$ and

$$|f'(x)| \leq x \quad \text{for all } 0 \leq x \leq 1.$$

Show that S has a compact closure in $C([0,1])$ with sup-norm $\|\cdot\|_\infty$, where

$$\|g\|_\infty = \sup_{0 \leq x \leq 1} |g(x)|.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable such that $\int_{\mathbb{R}} |xf(x)|\, dx < \infty$.

(a) Show that for each integer $n \geq 1$, the function $g_n(x) = f(x)\sin(x/n)$ belongs to $L^1(\mathbb{R})$.

(b) Find the limit

$$\lim_{n\to\infty} n \int_{\mathbb{R}} f(x) \sin\left(\frac{x}{n}\right) dx.$$

You must show all details.

3. Observe that $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ is a geometric series. Does the series converge in the L^2-norm on the interval $-1 < x < 1$? Explain.

4. (a) Suppose $f_n \to f$ in $L^2([0,1])$. Show that $f_n \to f$ in $L^1([0,1])$ as well.

(b) Find an example of a sequence $\{g_n\} \subset L^2([0,1])$ such that $g_n \to 0$ in $L^1([0,1])$ but $\{g_n\}$ does not converge to 0 in $L^2([0,1])$.

5. Let f belong to $C^\infty(\mathbb{R})$, that is, derivatives of all orders of f exist and are continuous on $\mathbb{R}$. Suppose that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) = 0$ (here, $f^{(n)}$ is the nth derivative of f). Show that there exists a non-empty open interval $I \subset \mathbb{R}$ and a polynomial P such that $f(x) = P(x)$ for all $x \in I$.

6. (a) Show that for any integer $k \geq 0$,

$$\lim_{n\to\infty} \int_0^1 x^k \sin(nx)\, dx = 0.$$

Suggestion: use integration by parts.

(b) Let f belong to $L^1([0,1])$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x) \sin(nx)\, dx = 0.$$

7. Let $f \in L^1([0,1])$ and set

$$\varphi(x) = \int_0^1 e^{xt} f(t)\, dt, \quad x \in \mathbb{R}.$$

Show that φ is differentiable on $\mathbb{R}$ and find a formula for φ'.

8. Let Q denote the set of all rational numbers in the interval $(0,1)$ and suppose that $I_1, \dots, I_N$ is a finite collection of open intervals which covers Q, i.e. $Q \subset \bigcup_{n=1}^{N} I_n$. Show that

$$1 \leq \sum_{n=1}^{N} \ell(I_n),$$

where $\ell(I)$ denotes the length of I. Is the same true if instead it is assumed that $I_1, \dots, I_n, \dots$ is an infinite collection of open intervals that covers Q? Explain.