Qualifying Examination in Topology

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If you believe there is an error on a question in this exam, report this to the proctor. If the proctor does not satisfactorily resolve your concern, you may modify the question so that in your view it is correctly stated but not in such a way that it becomes trivial.

Complete four questions from the eight questions in the first part and three questions from the six questions in the second part.

Section 1

1. (a) Suppose that A and B are nonempty sets and that $\pi : A \to B$ is surjective. Show there is an injective map $s : B \to A$ with the properties that $\pi \circ s = \mathrm{id}_B$. (b) Suppose that A and B are nonempty sets and $f : A \to B$ is injective. Show that there is a surjective map $r : B \to A$ with the property that $r \circ f = \mathrm{id}_A$.

2. Let Ω_1 be the set of ordinals less than or equal to the first uncountable ordinal with the order topology, and let $[0,1]$ have the order topology. Show that if $f : [0,1] \to \Omega_1$ is a continuous, then f cannot be onto.

3. Show that in a T_1 space the set of accumulation points of a subset is closed.

4. Suppose that X is a compact set and let $\{C_\alpha\}_{\alpha\in A}$ be a collection of nonempty close subsets that is closed under finite intersections. If U is an open set such that $\bigcap_{\alpha\in A} C_\alpha \subset U$, show that there is $\alpha_0 \in A$ such that $C_{\alpha_0} \subset U$.

5. Let X be Hausdorff and let $A \subset X$. If A is a retract of X, show that A is closed.

6. Show that if X is locally connected, that is, open connected sets are a neighborhood base, and $A \subset X$ is open and $B \subset A$ is closed in X, then A connected implies that $A - B$ is connected.

7. Let $\mathcal{A}$ be an index set and suppose that for each $\alpha \in \mathcal{A}$ there is an open continuous map $f_\alpha : X_\alpha \to Y_\alpha$ Show that if f_α is surjective for all but a finite number of α then $\prod_{\alpha\in\mathcal{A}} f_\alpha : \prod_{\alpha\in\mathcal{A}} X_\alpha \to \prod_{\alpha\in\mathcal{A}} Y_\alpha$ is an open continuous map.

8. Suppose that X is compact and Y is Hausdorff. Show that if $f : X \to Y$ is continuous then (a) f is closed (b) if f is surjective, then f is a quotient map (c) if f is bijective, then f is a homeomorphism

Section 2

1. (a) Let P^2 be two dimensional projective space, show that there does not exist a continuous map $s : P^2 \to S^2$ that satisfies $\pi \circ s = \mathrm{id}_{P^2}$, where π is the covering projection $\pi : S^2 \to P^2$. (b) If $S^1 \subset D^2$ is the boundary of the unit disk in the plane show that there does not exist a continuous map $r : D^2 \to S^1$ that satisfies $r \circ i = \mathrm{id}_{S^1}$ where $i : S^1 \to D^2$ is the inclusion.

2. Let $F : I \times I \to X$ be a continuous homotopy such that $F(s,0) = f(s)$ and $F(1,t) = l(t)$ Show that F is homotopic to $G : I \times I \to X$ such that $G(s,0) = f{*}l(s)$ and $G(1,t) = l(1)$.

3. Show that if $f : P^2 \to S^1$ is continuous then f is homotopic to a constant map.

4. Show that if $p : X \to Y$ is a covering projection and if any loop at $y_0 \in Y$ that lifts to a loop at $x_0 \in p^{-1}(\{y_0\})$ always lifts to a loop in X, then p is a regular cover, that is, $p_\sharp\pi_1(X, x_0)$ is a normal subgroup of $\pi_1(Y, y_0)$

5. Consider the $\mathbb{Z}_4$ action on $S^3 \subset \mathbb{C}^2$ generated by $\epsilon(z_1, z_2) = (iz_1, -iz_2)$ The quotient by orbits of this action is the lens space $L(4,3)$. If P^3 is three dimensional projective space show that the identity $\text{id} : S^3 \to S^3$ induces a double cover $p : P^3 \to L(4,3)$.

6 Let D be the unit disk in R^2 and let $k > 1$ be a fixed integer. Define an equivalence relation $\sim$ on D as follows. For $x, y \in \partial D$, $x \sim y$ if $y = e^{\frac{2\pi l}{k} i} x$ where $0 \leq l \leq k$ is an integer, and if $x \in D - \partial D$, then ,$x \sim x$. Let $M = D/\sim$. Compute $\pi_1(M)$