Algebra Qualifying Examination

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Instructions: Please do six problems, with three problem from each of the sections. Give complete proofs. If you attempt more than six problems, indicate clearly which six problems you would like to be graded. You have three hours to complete the exam.

Group theory

1. Prove that a group of order 182 is solvable. (Note that $182 = 2 \cdot 7 \cdot 13$.)
2. Prove that there are no simple groups of order $392 = 2^3 \cdot 7^2$.
3. Let G be a finite group and P a Sylow p-subgroup of G. If $N \trianglelefteq G$ prove that $P \cap N$ is a Sylow p-subgroup of N.
4. Let G be a group of real 2×2 matrices with determinant 1:

$$G = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \quad a, b, c, d \in \mathbb{R} \quad \text{such that} \quad ad - bc = 1 \}$$

   G acts on the complex upper half-plane $\mathbb{H} = \{z \mid Im(z) > 0\}$ by the rule $g \cdot z = \frac{az+b}{cz+d}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that the action is transitive and find the stabilizer at $z = i$.

Rings and Fields

1. Let R be a ring and M be an R-module. If $f : M \to M$ is an R-module homomorphism such that $f \circ f = f$, show that

$$M = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$$

2. Find all values of b in $\mathbb{Z}_5$ such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + x^2 + bx + b)$$

   is a field. Explain your answer.
3. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + 2$.
4. Show that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) = \mathbb{Q}(\sqrt{2} + \sqrt[3]{2})$