Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
Spring Semester, 2018

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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Suppose that $u(t), \alpha(t)$ are real-valued and continuous on $[a, b]$, and

$$u(t) \leq C + \int_a^t \big[\alpha(s)u(s) + K\big]ds, \quad a \leq t \leq b,$$

where $\alpha(t) \geq 0$ on $[a, b]$, C and K are nonnegative real numbers.

Prove:

$$u(t) \leq [C + K(t-a)]e^{\int_a^t \alpha(s)ds}, \quad a \leq t \leq b.$$

2. Consider the sequence of functions $\{y_k(t)\}_{k=0}^{\infty}$ defined by

$$\begin{cases} y_0(t) &= 2 - 3t, \\ y_{k+1}(t) &= 2 - \displaystyle\int_0^t (3 + \cos(y_k(\tau)))d\tau, \quad k = 0, 1, \ldots \end{cases}$$

Prove that $\{y_k(t)\}_{k=0}^{\infty}$ converges uniformly on the finite interval $[-N, N]$ where $N > 0$.

3. Consider the planar system

$$\begin{cases} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y). \end{cases}$$

Suppose that: **(H1)** $f \in C^1(\Omega)$ and $g \in C^1(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is simply connected; **(H2)** $f_x(x, y) + g_y(x, y)$ is positive almost everywhere in Ω.

Prove: the system has no closed orbit lying entirely in Ω.

4. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are continuous on $\mathbb{R}$. Prove that all the roots of $y_1(x)$ and $y_2(x)$ are simple and $y_1(x)$ has exactly one root between any two successive roots of $y_2(x)$.

5. Find the general solution of the differential equation

$$y''' + 3y'' + 3y' - 7y = 0$$

as a linear combination of three particular solutions $y_1(t), y_2(t), y_3(t)$. Prove that the three particular solutions are linearly independent. That is, show that if $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0$ for some constants c_1, c_2, c_3 and for all t, then $c_1 = c_2 = c_3 = 0$.

6. Solve the nonhomogeneous linear system for $x \in \mathbb{R}^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 2 - 2t \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

7. Consider the linear system of differential equations $dx/dt = Ax$. Suppose A is an $n \times n$ upper-triangular matrix with all its diagonal elements equal to one. Show that any solution equals e^t multiplied by a polynomial of t of order less than n.

Part II: Partial Differential Equations

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.
(a) If $v \in C^2(\Omega) \cap C(\overline{\Omega})$ and $-\Delta v \le 0$ in Ω. Prove that $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$.
(b) Prove that there exists a constant C depending only on Ω such that

$$\max_{\overline{\Omega}} |u| \le C(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|)$$

if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \le 0$ for $\lambda = \max_{\overline{\Omega}} |f|$.)

2. Find a solution to the heat equation with the initial values:

$$\begin{cases} u_t(x,t) - 9u_{xx}(x,t) = 0, & -\infty < x < \infty, \ t > 0; \\ u(x,0) = 4e^{-2x^2}, & -\infty < x < \infty. \end{cases}$$

3. Let $a > 0$ and let $\Omega \subset \mathbb{R}^n$ be a connected, bounded open set with a smooth boundary $\partial\Omega$. If $u(x,t)$ is a C^2 function on $\overline{\Omega} \times [0,\infty)$, and solves the initial/boundary-value problem

$$\begin{cases} u_{tt} - a^2 \Delta u = 0, & (x,t) \in \overline{\Omega} \times [0,\infty); \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty); \\ u(x,0) = 0, u_t(x,0) = 0, & x \in \overline{\Omega}. \end{cases}$$

Prove that u is identically equal to zero on $\overline{\Omega} \times [0,\infty)$.

4. Solve the first-order PDE:

$$u_x + yu_y + zu_z = u, \quad u(0,y,z) = y^2 + z^2.$$

5. Let $u(x,y)$ be a positive harmonic function on the disk

$$B_r(0) = \{(x,y) : x^2 + y^2 \le r^2\}.$$

(a) Prove that for any $\rho = \sqrt{x^2 + y^2} < r$,

$$\frac{r-\rho}{r+\rho} u(0) \leq u(x,y) \leq \frac{r+\rho}{r-\rho} u(0).$$

(b) Show that if $u(x,y)$ is a positive harmonic function in $\mathbb{R}^2$, then u is a constant function.

6. Consider the differential equation $u_{xx} + u_{yy} = 0$ with the boundary conditions $u(x,0) = 0$, $u_y(x,0) = g(x)$ in a neighborhood of the real line segment $\{(x,0) \mid -1 < x < 1\}$.

(a) Show that there is a unique solution if $g(x)$ is a real analytic function. Name the theorem you use.

(b) Show that if a solution exists, then $g(x)$ must be a real analytic function.

7. Let $f(x)$ be the odd, periodic function with period 2π satisfying $f(x) = x$ on $0 \leq x \leq \pi/2$ and $f(x) = (\pi - x)$ on $\pi/2 \leq x \leq \pi$. Find the solution of the initial value problem for the wave equation.

$$\begin{cases} u_{tt}(x,t) - 4u_{xx}(x,t) = 0, & -\infty < x < \infty, t > 0; \\ u(x,0) = f(x), & -\infty < x < \infty; \\ u_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$