Real Analysis, Ph.D. Qualifying Exam

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Instructions: Do six of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours. In this exam, Lebesgue measure on $\mathbb{R}$ or on any interval is denoted by m.

1. Let F be a compact set in $\mathbb{R}$. For each integer $n \geq 1$, define

$$V_n = \bigcup_{x \in F} \left(x - \frac{1}{n},\, x + \frac{1}{n} \right).$$

Show that

$$\lim_{n \to \infty} m(V_n) = m(F).$$

2. (a) State the Lebesgue Dominated Convergence Theorem.

(b) Prove that there does **not** exist a function $f \in L^1([0,1])$ such that for any integer $n \geq 1$,

$$f(x) \geq n^2 (1-x) x^n \quad \text{for all} \quad x \in [0,1]?$$

3. Let S be the set of all functions f that are continuous on $[0,1]$ and differentiable on $(0,1)$ such that $f(0) = 0$ and $|f'(x) + f(x)| \leq 1$ for all $x \in (0,1)$. Show that S has a compact closure in $C([0,1])$ with sup-norm $\|\cdot\|_\infty$, where

$$\|g\|_\infty = \sup_{0 \leq x \leq 1} |g(x)|.$$

(Hint: for a differentiable function f on $(0,1)$, what is the derivative of $h(x) = f(x)e^x$?)

4. Give an example of a sequence of functions $f_n \in L^1([0,1])$ for all $n = 1, 2, \ldots$ and a function $g \in L^1([0,1])$ with the following properties:

(a) $f_n(x) \to g(x)$ for almost all $x \in [0,1]$;

(b) $\displaystyle\int_{[0,1]} |f_n| \, dm = 2$ for every $n = 1, 2, \ldots$;

(c) $\displaystyle\int_{[0,1]} |g| \, dm = 1$.

5. Let $E \subset \mathbb{R}$ be measurable with a finite measure. Let $\{f_n\}$ be a sequence of measurable real-valued functions on E that converges to a function f pointwise on E. Show that $E = \cup_{k=1}^{\infty} E_k$, where for each index k, the set E_k is measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.

6. (a) State Hölder's inequality.

   (b) Let f be a function in $L^1([0, 2\pi])$. Show that

$$\left(\int_{[0,2\pi]} f(x) \sin(x) \, dm(x) \right)^2 + \left(\int_{[0,2\pi]} f(x) \cos(x) \, dm(x) \right)^2 \le \left(\int_{[0,1]} |f(x)| \, dm(x) \right)^2.$$

7. (a) State the Stone-Weierstrass Theorem.

   (b) Show that the linear span of $\left\{ x^j e^{ky} : j = 0, 1, 2, \dots ; k = 0, 1, 2, \dots \right\}$ is dense in $C([0,1] \times [0,1])$, the space of real-valued continuous functions on $[0,1] \times [0,1]$.

   (c) Show that the span of $\left\{ x^j e^{jy} : j = 0, 1, 2, \dots \right\}$ is **not** dense in $C([0,1] \times [0,1])$.

8. Let f be a continuous function on $[0,1]$. For any integer $k \ge 1$, let f_k be the step function defined on $[0,1]$ by $f_k(0) = 0$ and

$$f_k(x) = k \int_{\frac{j}{k}}^{\frac{j+1}{k}} f(t) \, dt, \quad \text{for} \quad \frac{j}{k} < x \le \frac{j+1}{k} \text{ with } 0 \le j \le k-1.$$

   Show that f_k converges to f in L^1-norm as $k \to \infty$.