Algebra Qualifying Examination

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Instructions: Please do six problems, with three problem from each of the sections. Give complete proofs. If you attempt more than six problems, indicate clearly which six problems you would like to be graded. You have three hours to complete the exam.

Group theory

1. Let G be an abelian group and let $n > 0$ be an integer. Prove that $\{g^n \mid g \in G\}$ is a subgroup of G. Show that this is not necessarily the case for nonabelian groups.
2. Show that a group of order 40 has exactly 4 elements of order 5.
3. Let H and K be normal subgroups of G with $H \cap K = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$.
4. Let G be a group with an even number of elements. Let e denote the identity of G. Show that there is an element $a \in G$ such that $a \neq e$ and $a.a = e$

Rings and Fields

1. Let R be a ring containing $\mathbb{C}$ as a subring. Prove that there are no ring homomorphism $R \to \mathbb{R}$.
2. (a) Let R be a commutative ring with 1. Show that if M is a maximal ideal of R then M is a prime ideal of R.

   (b) Give an example of a non-zero prime ideal in a ring R that is not a maximal ideal.
3. Show that x^6+x^3+1 is irreducible over $\mathbb{Q}$. Describe the splitting field of x^6+x^3+1 over $\mathbb{Q}$ and find its degree over $\mathbb{Q}$.
4. Let α, β be transcendental over $\mathbb{Q}$. Prove that either $\alpha\beta$ or $\alpha + \beta$ is also transcendental.