Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

April 13, 2019

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. (25 pts) Suppose $\{X_i, i \geq 1\}$ are independent r.v.'s such that

$$\begin{aligned} P(X_i = -1) &= P(X_i = 1) = \frac{1-2^{-i}}{2}, \\ P\left(X_i = -2^i\right) &= P\left(X_i = 2^i\right) = \frac{2^{-i}}{2} \end{aligned}$$

Define $S_n = \sum_{i=1}^{n} X_i$ and show that
a. (10 pts) $X_i/i \to 0$ a.s..
b. (15 pts) $\sum_{i=1}^{n} X_i/i$ converge a.s..

2. (25 pts) Let $X_1, ..., X_n \sim_{iid} E(0, \theta)$ with the density function $\theta^{-1} e^{-x/\theta} I_{(0,\infty)}(x)$ with $\theta > 0$. The prior of $y = \theta^{-1}$ is the gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$ with density function $\frac{1}{\Gamma(\alpha)\gamma^{\alpha}} y^{\alpha-1} e^{-y/\gamma} I_{(0,\infty)}(y)$.
a. (10 pts) Find the posterior distribution of θ^{-1} given $X_1, ..., X_n$.
b. (10 pts) Find the Bayes action $\delta_t(X_1, \cdots, X_n)$ of $\eta(t) = e^{-ty}$ with fixed $t \neq 0$ under squared error loss.
c. (5 pts) Show that the Bayes action $\delta_t(X_1, \cdots, X_n)$ is consistent.

3. [25 points] Let $X_1, X_2, \ldots$ be independent identically distributed random variables with common mean μ and finite variance σ^2. Write

$$T_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} X_i X_j.$$

(a) Show that $T_n \xrightarrow{p} \mu^2$ as $n \to \infty$, where the symbol $\xrightarrow{p}$ denotes convergence in probability.

(b) Show that T_n is asymptotically normal: there is a constant σ_T^2 such that $\sqrt{n}(T_n - \mu^2) \xrightarrow{d} N(0, \sigma_T^2)$ as $n \to \infty$, where the symbol $\xrightarrow{d}$ denotes convergence in distribution.

(c) Find σ_T^2 in part (b).

4. [25 points] Consider Bayesian estimation in which the parameter θ has a standard exponential distribution, so the prior density of θ is $\pi(\theta) = e^{-\theta}$, $\theta > 0$, and given θ, $X_1, \ldots, X_n$ are i.i.d. from an exponential distribution with failure rate θ, so $p(x|\theta) = \theta e^{-\theta x}$ for $x > 0$ and $\theta > 0$. Determine the Bayes estimator of θ if the loss function is $L(\theta, d) = (d - \theta)^2/d$.