Real Analysis, Ph.D. Qualifying Exam

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Instructions: Do six of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours. In this exam, Lebesgue measure on $\mathbb{R}$ or on any interval is denoted by m.

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous for each n. Suppose that for each $x \in [0,1]$, the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Show that there exist a non-empty open interval $I \subset [0,1]$ and a constant $M > 0$ such that

$$\sum_{n=1}^{\infty} |f_n(x)| \leq M \quad \text{for all } x \in I.$$

2. (a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$. Define

$$f(x) = \inf_{n \geq 1} f_n(x) \text{ for all } x \in \mathbb{R}.$$

Show that f is also Lebesgue measurable.

(b) Assuming the Monotone Convergence Theorem, prove Fatou's Lemma.

(c) Assuming Fatou's Lemma, prove the Monotone Convergence Theorem.

3. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable for each n such that $\|f_n\|_3 \to 0$ and $\|f_n\|_5 \to 0$ as $n \to \infty$. Prove or give a counterexample for each of the statements below.

(a) $\|f_n\|_2 \to 0$.

(b) $\|f_n\|_4 \to 0$.

4. Let f be a continuous function on $[0,1]$. Find the limit

$$\lim_{\epsilon \to 0^+} \epsilon \int_{[0,1]} x^{\epsilon - 1} f(x)\, dx.$$

Suggestion: consider $f(x)$ is a polynomial first.

5. Let $\mathcal{F}$ be the collection of all continuous functions on $[0,1]$ that can be represented in the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx),$$

where $|a_n| \leq 2^{-n}$ for all $n \geq 0$. Show that $\mathcal{F}$ has a compact closure in $C([0,1])$ with max norm.

6. Show that the following limit exists and find the limit.

$$\lim_{n\to\infty} \int_{(0,\infty)} \frac{\cos(x^n)}{1+x^n}\, dm(x).$$

7. Let $\phi \in L^\infty(0,\infty)$. Show that

$$\lim_{n\to\infty} \left[\int_{(0,\infty)} |\phi(x)|^n e^{-x}\, dm(x) \right]^{1/n} = \|\phi\|_\infty.$$

8. We say that functions f_n, for $n = 1, 2, \ldots$, integrable on a measurable set A are **equi-integrable** if for every $\epsilon > 0$ there is $\delta > 0$ such that for every measurable subset B of A, we have $\int_B |f_n| dm < \epsilon$ for $n = 1, 2, \ldots$ whenever $m(B) < \delta$. Show that if $\{f_n\}$ is a pointwise convergent sequence of equi-integrable functions on a set A of finite measure, then

$$\lim_{n\to\infty} \int_A f_n\, dm = \int_A (\lim_{n\to\infty} f_n)\, dm.$$

Suggestion: use Egorov's Theorem.