PhD Qualifying Examination in Topology

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If you believe there is an error on a question in this exam, report this to the proctor. If the proctor does not satisfactorily resolve your concern, you may modify the question so that in your view it is correctly stated but not in such a way that it becomes trivial.

Answer three questions of the following six questions.

Section 1

1. Let D^2 be the unit disk in $\mathbb{R}^2$ and let $\varphi : I \times I \to D^2$ given by $\varphi(s,t) = (s\cos(2\pi t), s\sin(2\pi t))$. Show that φ is a quotient map.
2. Two elements of a topological space X are q-related if and only if for any separation of X both elements belong to the same open set. Show that q-relatedness is an equivalence relation, and show that the equivalence classes are closed subsets.
3. Recall that a collection of subsets $\mathcal{C}$ of a topological space X is neighborhood finite if for any $x \in X$, there is a neighborhood of x that intersects only finitely may elements of $\mathcal{C}$. Show that if $\{U_\alpha\}_{\alpha\in A}$ is a neighborhood finite family and if $B \subset A$, then $\overline{\bigcup_{\beta\in B} U_\beta} = \bigcup_{\beta\in B} \overline{U_\beta}$.
4. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of locally compact topological spaces, and let $X = \prod_{\alpha\in A} X_\alpha$ be the product space. Show that X is locally compact if and only if X_α is compact for all but a finite number of $\alpha \in A$. If needed you may assume the Tychonoff Theorem.
5. A topological space X is T_5 if for any separated subsets A and B of X, there are disjoint open neighborhoods of U_A and U_B of A and B respectively. Recall that $A, B \subset X$ are separated if and only if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Show that a space is T_5 if and only if every subspace is T_4.
6. A sequence $S = \{s_i\}_{i=0}^\infty$ in a topological space X is said to have a limit point $x \in X$, if every neighborhood of x contains infinitely many elements of S. Show that if X has a sequence that has no limit points, then X is not compact.

Answer three questions of the following six questions.

Section 2

1. Consider the comb space $Y = \{(s,t) \in I \times I | t = 0 \text{ or } s \in \{1/n\}_{n=1}^\infty \cup \{0\}\} \subset \mathbb{R}^2$. Show that the constant map $c : Y \to Y$ given by $c(s,t) = (0,1)$ is not homotopic to the identity relative to $\{(0,1)\}$.
2. Let $f, g : S^1 \to (X, x_0)$ be continuous base point preserving maps. View S^1 as the quotient $S^1 = [0,1]/\sim$, where $0 \sim 1$, and let $[0]$ be the base point. Recall that the concatenation of f and g is defined by
$$f * g(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$
Construct a homotopy to show that $f * (g * h)$ is homotopic to $(f * g) * h$.
3. Suppose that Y is path connected and X is path connected and locally path connected and that $p : (Y, y_0) \to (X, x_0)$ is a covering projection that is not a homeomorphism. Use the homotopy lifting property to show that $\pi_1(X, x_0)$ is nontrivial.
4. Suppose that Y is path connected and X is path connected and locally path connected and that $p : (Y, y_0) \to (X, x_0)$ is a covering projection. Show that if for any $y \in p^{-1}(x_0)$, $p_\#\pi_1(Y, y) = p_\#\pi_1(Y, y_0)$, then $p_\#\pi_1(Y, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$.

5. Let a group G act properly discontinuously on a pair of path connected, locally path connected Hausdorff spaces X and Y, and let $\varphi : X \to Y$ be an equivariant map, that is for any $g \in G$ and $x \in X$, $\varphi(g \cdot x) = g \cdot \varphi(x)$. If $p_X : X \to X/G$ and $p_Y : Y \to Y/G$ are the covering projections, show that φ induces an isomorphism,

$$\widetilde{\varphi}_{\#} : \frac{\pi_1(X/G, p_X(x))}{p_{X\#}\pi_1(X, x)} \to \frac{\pi_1(Y/G, p_Y(\varphi(x)))}{p_{Y\#}\pi_1(Y, \varphi(x))}.$$

6. Let P^2 be the real two dimensional projective space realized as $P^2 = I \times I/\sim$ where the nontrivial equivalences are $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (1-x, 1)$. Use the Seifert Van Kampen Theorem to compute the fundamental group of $P^2 - \{[(1/2, 1/2)]\}$.