Mathematics Department March 29, 1997

Ph.D. Qualifying Exam

ALGEBRA

March 29, 1997

INSTRUCTIONS: Do any four problems. And no more than four.

Please make sure that you give complete solutions to each problem that you do.

Indicate which problems you wish to have graded.

You have three hours.

POLICY ON MISPRINTS

The Ph.D. Qualifying Examination Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

1. Let G be a finite group, p a prime, and P a p-subgroup of G. Let n be the number of Sylow p-subgroups of G that contain P.

   (a) [4 points] Prove $n \equiv 1 \bmod p$.

   (b) [4 points] Prove $n \geq |Syl_p(N_G(P)|$.

   (c) [2 points] Prove that $G = S_4$ (the symmetric group on 4 letters) is an example of strict inequality in part (b).

2. Let G be a finite simple group of order 168.

   (a) [2 points] Show that G has precisely 8 Sylow 7-subgroups.

   (b) [3 points] Show that G is isomorphic to a subgroup of $\tilde{G}$ of A_8 and that no element of order 2 in $\tilde{G}$ has a fixed point.

   (c) [2 points] Show that G has no element of order 6.

   (d) [3 points] Find the number of Sylow 3-subgroups of G.

3. Find an orthogonal matrix Q and a diagonal matrix Λ so that $Q^{-1}AQ = \Lambda$ where

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

4. Let R be a ring and M be an R-module of finite composition length. If f is an endomorphism of the R-module M, show that there is an integer k such that $M = \text{Im } f^k \oplus \text{ Ker } f^k$.

5. If $K = \mathbb{Q}(\sqrt{a})$, where a is a negative integer and $\mathbb{Q}$ is the rationals, show that K can't be embedded in a cyclic extension whose degree over $\mathbf{Q}$ is divisible by 4.

6. Let p be a prime and let $GF(p^m)$ denote a finite field of order p^m.

   (a) [5 points] Show that $GF(p^m)$ is isomorphic to a subfield of $GF(p^n)$ if and only if m divides n.

   (b) [5 points] Let E be the algebraic closure of $GF(p)$. Show that there is an intermediate field L between $GF(p)$ and E with $|L : GF(p)| = \infty$ and

$|E : L| = \infty$.

7. Let R be a commutative ring with 1 and $R[x]$ the ring of polynomials in one (commuting) indeterminate with coefficients in R.

   (a) [8 points] For each of the following statements indicate whether it is true or false. If it is false, give a counterexample. If it is true you do NOT have to provide a proof.

   (i) If R is a PID then so is $R[x]$.

   (ii) If R is a UFD then so is $R[x]$.

   (iii) If R is artinian then so is $R[x]$.

   (iv) If R is noetherian then so is $R[x]$.

   (b) [2 points] What are the units in $R[x]$? Justify your answer.

8. Let R be a ring with 1. If M is an R-module, the **uniform dimension** of M (ud M) is the largest integer n such that there is a direct sum $M_1 \oplus \ldots \oplus M_n \subseteq M$ with all the M_i non-zero. If no such integer exists then we say ud $M = \infty$. If $M \subseteq N$ are R-modules, M is said to be **essential** in N if every non-zero submodule of N has non-zero intersection with M. Suppose the ud $M < \infty$ and $M \subseteq N$. Prove that M is essential in N if and only if ud $M =$ ud n.