.

TOPOLOGY QUALIFYING EXAMINATION

Time: Three hours.

Do FOUR problems in each section.

April 1997

Section A

1. Let $A_1, A_2, \ldots, A_k$ be a collection of connected subsets of a topological space X. Suppose that each A_i is *not* disjoint from all the remaining A_i's. Show that $Y = A_1 \cup A_2 \ldots \cup A_k$ is connected.

2. Let $\mathbb{R}$ have the "finite-closed" topology i.e., the closed sets other than $\mathbb{R}$ and $\emptyset$ are the finite subsets of $\mathbb{R}$. Is this space:

   (a) connected?

   (b) Hausdorff?

   (c) metrizable?

   (d) Also describe the compact subspaces of this space.
   Justify your answer in each case.

3. (a) Let X be Hausdorff topological space. Show that X is normal iff each neighborhood of a closed set F contains the closure of some neighborhood of F.

   (b) Let A be an indexing set and let X be a topological space. Define the disjoint union of subsets X_α of X to be $\cup(\alpha \times X_\alpha)$ where $\alpha \in A$ and $A \times X$ is topologized by taking a basis of open sets to be $\alpha \times U_\alpha$ where U_α is open in X_α. Suppose that the X_α are closed in $\cup X_\alpha (\subset X)$ and A is finite. Show that $j : \cup(\alpha \times X_\alpha) \longrightarrow \bigcup_{\alpha \in A} X_\alpha$ is an identification map.

4. Let $p : X \longrightarrow Y$ be a continuous map of Hausdorff spaces. Suppose that for any $y \in Y, p^{-1}(y)$ is compact and second countable and there exists an open neighborhood U_y such that $p^{-1}(U_y)$ is homeomorphic to $p^{-1}(y) \times U_y$. Prove that p is a closed map.

5. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. Define the *graph* of f to be the subset $\{(x, f(x) : x \in X\}$ of $X \times Y$. Prove that:

   (a) if Y is Hausdorff then the graph of f is closed.

   (b) if X is connected then the graph of f is connected.

Section B

1. .

   (a) Compute the fundamental group of the space obtained by identifying the points x and y pictured above on the *solid* three hole torus.

   (b) Compute the fundamental group of the space obtained by identifying the points x and y on the *surface* of the three hole torus.

2. (a) Define the join (or product) of two compatible loops α, β in a topological space X. If γ is a third loop find an explicit homotopy between the loops $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$.

   (b) The space G is a topological group meaning that G is a group and also a Hausdorff topological space such that the multiplication and map taking each element to its inverse are continuous operations. Given two loops based at the identity e in G, say $\alpha(s)$ and $\beta(s)$, we have two ways to combine them: $\alpha \cdot \beta$ (join of loops as in (a)) and secondly $\alpha\beta$ using the group multiplication. Show, however, that these constructions give homotopic loops.

3. .

   The annulus shown has its edges identified to form a surface S. [“Annulus” means we start with the shaded region i.e., the interior of the inner rectangle is ignored].

Identify S as one of the standard closed surfaces in the following two ways:

(a) By cutting along PQ, represent S as a polygon with edges identified in pairs.

(b) By cutting around the circle γ.

4. Let S^1 denote the unit circle with center the origin in $\mathbb{R}^2$.

(a) Define $f : S^1 \longrightarrow S^1$ by $f(x) = -x$. Prove that f is homotopic to the identity map.

(b) Let $g : S^1 \longrightarrow S^1$ be a map which is not homotopic to the identity. Show that $g(x) = -x$ for some x in S^1.

5. Let $\pi : \tilde{X} \longrightarrow X$ be a covering space.

(a) Show that π is an open map.

(b) Show that if $\tilde{X}$ is homeomorphic to $X \times \pi^{-1}(p)$, where $\pi(q) = p$ with q, p being base points in $\tilde{X}$ and X, respectively, there is a continuous map $s : X \longrightarrow \tilde{X}$ such that $\pi \circ s =$ identity on X.

(c) Assume now that X is path-connected. Show that if there exists a continuous map $s : X \longrightarrow \tilde{X}$ such that $\pi \circ s =$ identity on X, then $\tilde{X}$ is disconnected unless $\pi^{-1}(p)$ consists precisely of q.