Real Analysis Qualifying Exam
January, 1999
Closed book, no notes.
Justify your answer with as much detail as possible.

Do any six (6) of the following eight (8) problems

1. Let $\mathcal{P}^n$ be the set of all $(p_1, p_2, \ldots, p_n)$ such that each $p_i \geq 0$, and $\sum p_i = 1$, and let ln denote the natural logarithm.

   (a) Show that $\ln x \leq x - 1$, for $x > 0$, with equality only when $x = 1$.

   (b) Show that $\sum_{i=1}^{n} p_i \ln \frac{q_i}{p_i} \leq 0, (p_1, p_2, \ldots, p_n), (q_1, q_2, \ldots, q_n) \in \mathcal{P}^n$, with equality if and only if $(p_1, p_2, \ldots, p_n) = (q_1, q_2, \ldots, q_n)$.

2. Let $X = \{0, 1\}^\infty$ be the set of all sequences $x = (x_1, x_2, \ldots)$ of 0's and 1's.

   (a) Let $Y \subseteq X$ consist of those $x \in X$ that are eventually 0, that is, for which there is an $N(x)$ such that $x_n = 0, n \geq N(x)$. Show that Y is countable.

   (b) Show that X is not countable.

   (c) Show that X is in one-to-one correspondence with the unit interval.

3. Suppose $\{f_n\}$ and $\{g_n\}$ are sequences of real-valued functions defined on a set E such that

   (a) $\sum f_n$ has uniformly bounded partial sums.

   (b) $g_n \to 0$ uniformly on E.

   (c) $g_n(x) \geq g_{n+1}(x), x \in E, n = 1, 2, \ldots$.

   Prove that $\sum f_n g_n$ converges uniformly on E.

4. Let f be a real valued function defined on an interval $[a, b]$. For any sub-interval I, we define the oscillation of f over I as follows:

$$\omega_f(I) = \sup_{x,y \in I} |f(x) - f(y)|.$$

Further we define the oscillation at a point x as follows:

$$\omega_f(x) = \lim_{h \to 0} \omega_f([x - h, x + h]).$$

(a) Show that f is continuous at x if and only if $\omega_f(x) = 0$.

(b) For each positive constant c, let E_c be the set of x such that $\omega_f(x) \geq c$. Show that E_c is a closed set.

(c) We say that f is Riemann Integrable over $[a, b]$ if given any $\epsilon > 0$, there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that

$$\sum_{i=1}^{n} \omega_f([x_{i-1}, x_i])|x_{i-1} - x - i| < \epsilon.$$

Show directly that the measure of E_c is 0 for every c. (Here directly means without using the the theorem that states that the set of discotinuities of a Riemann integrable function has Lebesgue measure 0.)

5. We know from the binomial theorem that

$$1 = \sum_{r=0}^{n} \binom{n}{r} x^r (1 - x)^{n-r}.$$

Show that the following identities are true.

(a) $nx = \sum_{r=0}^{n} r \binom{n}{r} x^r (1 - x)^{n-r}$.

(b) $n(n - 1)x^2 + nx = \sum_{r=0}^{n} r^2 \binom{n}{r} x^r (1 - x)^{n-r}$.

6. Given a sequence $\{f_n\}$ of measurable functions, let E be the set of points x for which $\lim_{n\to\infty} f_n(x)$ exists. Prove that E is measurable.

7. Gauss' second mean value theorem is normally stated as follows: Assume f and g are Riemann Integrable on $[a, b]$ and g is monotone. Then there exists a ξ in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = g(a)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x0dx.$$

Proof of this in this generality is quite involved.

   (a) But assuming that g is continuously differentiable, show the truth of Gauss' theorem.

   (b) If g is monotone decreasing and $g \geq 0$, show that for some $\xi \in [a, b]$,

$$\int_a^b f(x)g(x)dx = g(a)\int_a^\xi f(x)dx.$$

8. State Stone-Weierstrass theorem and using this or otherwise show that any continuous function of period 2π can be uniformly approximated by trigonometric polynomials i.e., linear combinations of the functions $\cos nx, \sin nx, n \geq 0$.