TOPOLOGY QUALIFYING EXAM

Do four problems in each section. Explain your arguments clearly and carefully.

Section 1

1. Prove or disprove: if $f : [a,b] \longrightarrow [c,d]$ is continuous, surjective, and increasing then f is a homeomorphism.

2. Prove or disprove: in a compact topological space every infinite set has a limit point.

3. Let A be a subspace of a topological space X. Prove A is disconnected if and only if there exist open subsets P and Q of X such that $A \subset P \cup Q$, $P \cap Q \subset X - A$ and $P \cap A \neq \emptyset$, $Q \cap A \neq \emptyset$. [*Note:* $X - A$ denotes the complement of A in X.]

4. Prove that if $f : X \longrightarrow Y$ is continuous and surjective, X is compact and Y is Hausdorff then f is an identification map.

5. Let A and B be disjoint compact subsets of a Hausdorff space X. Then there are disjoint, open sets containing A and B. Prove this theorem.

Section 2

1. A connected topological space X is said to be *contractible* if the identity map of X to itself is homotopic to a constant map on X. Prove that X is contractible if and only if X has the homotopy type of a point.

2. Compute the fundamental groups of the following spaces X by any valid method.

   i. X is a tube with *closed* ends and two points are removed.

   ii. X is a tube with *open* ends and two points are removed.

3. Prove that the closed unit disk D in $\mathbb{R}^2$ cannot be retracted to the unit circle S^1. Deduce that any continuous map $f : D \longrightarrow D$ has a fixed point. [*Hint:* consider the line joining x to $f(x)$ where $x \in D$.]

4. Identify the surface whose polygonal symbol is the given octagon. What is the surface's fundamental group?

5. i. Let $f : (0, 17) \longrightarrow S^1$ by $f(x) = e^{2\pi i x}$. Is f a covering map?

   ii. Describe all covering spaces of the Klein bottle and justify that they are the only ones.