University of Toledo, Department of Physics and Astronomy

Ph.D. Qualifying Exam

Fall 2021
October 16

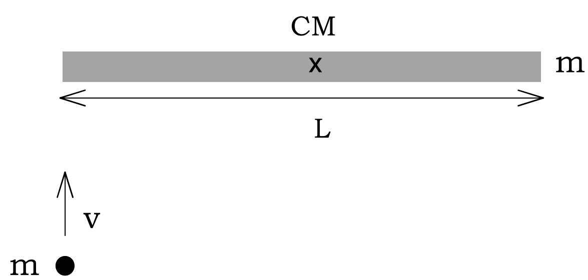
Instructions:

- **Do not** write your name on your exam; put your chosen letter on every sheet of paper that you turn in.
- Work 2 out of 3 problems in each category.
- Begin each problem on a new sheet of paper.
- Be sure to state which problems are omitted.

CLASSICAL MECHANICS

1. A ring of mass *m* and radius R is suspended under the influence of gravity from a point on its circumference.
   a. Assuming that all motions are in the plane of the ring, write down the Lagrangian as well as the dynamical equation of motion.
   b. Use these results to determine the period of oscillation in the small-angle approximation.

2. A particle of mass *m* and speed v undergoes an elastic collision with a rod of mass m and length L which is initially at rest as shown in the diagram below. Determine the speed v_1 of the particle, speed v_2 of the center-of-mass (CM) of the rod, and angular velocity ω of rotation about the CM of the rod after the collision. (Note: you can assume that both the particle and rod slide freely on a horizontal surface without friction.)



3. A point mass *m* is constrained to move on a massless hoop of radius R fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed ω. Obtain the Lagrange equations of motion assuming that the only external forces arise from gravity. Show that if $\omega > \omega_0$ (where ω_0 is a critical value) there exists a solution in which the particle remains stationary on the hoop at a point other than the bottom of the hoop, but if $\omega < \omega_0$ the only stationary point for the particle is at the bottom of the hoop. What is the value of ω_0?

ELECTRICITY & MAGNETISM

1. A charge density

$$\rho = \rho_0 \left(\frac{r}{a}\right)$$

is placed inside a grounded conducting spherical shell of radius a.

a. Find the total charge q_a induced on the inner surface ($r = a$) of the conducting shell.

b. Find the electric field $\boldsymbol{E}(r)$ inside the shell.

c. What is the electrostatic potential $\Phi(r)$ inside the shell?

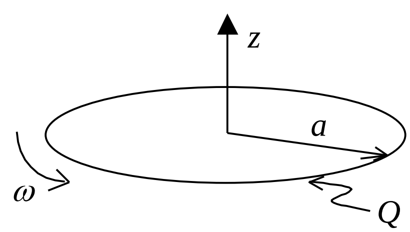
2. Consider a "point" dipole with dipole moment $p = p\hat{z}$ placed at the origin of a grounded conducting spherical shell of radius a. Note: a point dipole is one in which the charge separation vanishes while the dipole strength p is held fixed.

a. First ignore the conducting shell, and write down an expression for the electrostatic potential $\Phi(r, \theta)$ at small distances ($r \ll a$).

b. Now include the spherical conducting shell and solve for the electrostatic potential at all positions within the interior of the sphere ($0 < r < a$). Hint: In the limit of small r, your expression should match the answer for part (a).

c. Find the surface charge density $\sigma(\theta)$ induced on the interior surface of the spherical shell.

3. A non-conducting ring of radius a has a total charge Q distributed uniformly along its circumference. The ring rotates about its symmetry axis with an angular frequency ω



a. What is the current I carried by the ring?

b. What is the magnetic field **B** (direction and magnitude) at the center of the ring?

c. What is the magnetic field **B** at very large distances $r \gg a$? Note: The magnetic vector potential for a point dipole at the origin is

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \ .$$

QUANTUM MECHANICS

1. A quantum particle of mass M moves in one-dimensional quartic potential $U(x) = Cx^4$ where C is a positive constant.
   Derive approximate expressions for its energy spectrum
   a. From the dimensional analysis
   b. Using de Broglie quantum mechanics (i. e. an integral number of de Broglie waves along the particle trajectory).

2. A particle in a box [$V(x)=0,\ 0\le x\le a;\ V=\infty,\ x<0$ or $x>a$] is prepared in a state characterized by a wave function

$$\psi(x) = \begin{cases} Cx(a-x) & 0 \le x \le a \\ 0 & x < 0; x > a \end{cases}$$

   a. Find the normalization constant C from the physical interpretation of a wave function.
   b. What is the probability for finding the particle in the region of [0, $a/4$]?
   c. What are the average position and the uncertainty of position in this state?
   d. What are the average momentum and the uncertainty of momentum in this state?

   Verify that the uncertainty principle is satisfied.

3. For the electron in the ground state of the lithium ion Li^{2+},
   a. Prove that the wave function has the form

$$\psi(r) = Aexp(-r/a), \quad \text{with } A,a = \text{const.}$$

   b. Calculate A and a.
   c. Prove that the mean value of $1/r$ is $1/a$.
   d. Calculate the average kinetic energy.

   (Hint: the radial component of the Laplacian can be written as $\frac{1}{r}\ \frac{d^2}{dr^2}\ r$ where r is the radial distance.)