University of Toledo, Department of Physics and Astronomy

Ph.D. Qualifying Exam

Fall 2017
September 30

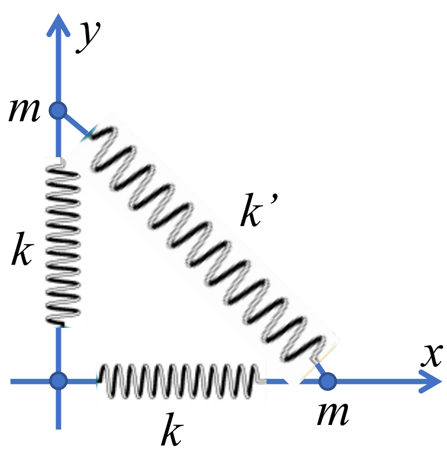
Instructions:

- **Do not** write your name on your exam. On every sheet of paper that you turn in, put your assigned letter code at the upper left corner and a label identifying the problem (such CM1, EM2, QM3) at the upper right corner.
- Work 2 out of 3 problems on each subject. Staple together solutions for each subject and turn them in separately. Indicate clearly which problems are to be graded and which are to be omitted.
- Begin each problem on a new sheet of paper.

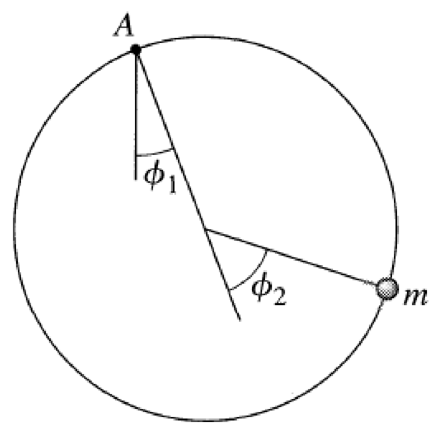
CLASSICAL MECHANICS (CM)

1. A particle with a mass of m doing one-dimensional motion is under the influence of a force $F(x) = -kx + kx^3/a^2$, where k and a are positive constants. If the particle starts at $x=0$ with velocity $v_0 = a\sqrt{\frac{k}{2m}}$ going positive x direction, find the time for the particle to reach $x = a/2$.

2. Two equal masses m are constrained to move without friction, one on the positive x axis, and the other one on the positive y axis. They are attached to two identical springs (with force constant of k) whose other ends are attached to the origin. In addition, the two masses are connected to each other by a third spring of force constant k'. The springs are chosen so that the system is in equilibrium with all three springs relaxed. What are the normal frequencies of this system when it undergoes a small oscillation?



(For Problem 2)

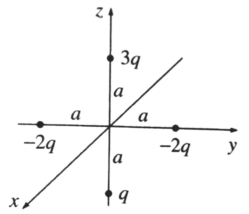


(For Problem 3)

3. A bead of mass m is threaded on a frictionless circular wire hoop of radius R and mass of M. The hoop is suspended at the point A and is free to swing in its own vertical plane as shown. Using angle ϕ_1 and ϕ_2 as generalized coordinates, solve for the normal frequencies of small oscillations and find the motion in the corresponding normal modes.
To simplify the calculations, you can assume that the hoop mass M = m also.

ELECTRICITY AND MAGNETISM (EM)

1. The figure shows a series of charges held relative to each other by insulating sticks.
(a) Calculate the dipole moment in terms of a and q. Remember this is a vector!

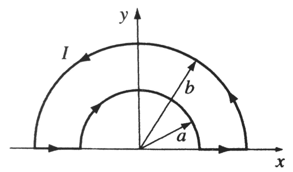


(b) The dipole is placed in a electric field $\vec{E} = E_0\hat{x} + E_0\hat{z}$. What is the torque in terms of E_0, q and a? In what direction is the torque vector? In what direction will the dipole rotate?

2. A piece of wire, bent into a loop, carries a current that increases linearly with time given by $I(t) = kt$ over the range $-\infty < t < \infty$. We can write the $\vec{A}$ field at the origin by the equation

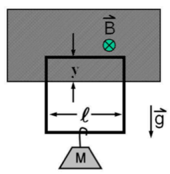
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(t_r)}{\mathscr{r}}dl = \frac{\mu_0 k}{4\pi}\int \frac{(t - \mathscr{r}/c)}{\mathscr{r}}d\vec{l} = \frac{\mu_0 k}{4\pi}t\int \frac{d\vec{l}}{\mathscr{r}} - \frac{1}{c}\int d\vec{l}$$

where $\int d\vec{l} = 0$ and the integral is performed over the wire shown in the figure.



Calculate the value of the vector potential $\vec{A}$ at the origin in terms of k, a, b, and t and then determine the electric field at the center (assuming the electric potential is constant: $V = const.$).

3. A rectangular wire loop of length ℓ is placed in a uniform magnetic field $\vec{B}$ pointed into the page. Assume the width of the loop in the field is y. A battery maintains a current I in the loop and a mass M is attached to the bottom of the loop. The loop is in the Earth's gravitational field and define $\hat{y}$ downward such that the direction of the gravitational pull $\vec{g} = g\hat{y}$ to avoid extra minus signs later on. Assume that the wire has a resistance R. Assume that the battery and loop have negligible mass compared to M.



(a) Find the current necessary to maintain the loop in static equilibrium. Remember that the Lorenz force for a current is $\vec{F} = \int I(d\vec{l} \times \vec{B})$.

(b) The external current in the wire is now suddenly turned off (i.e., the battery is shorted across its terminals). Find the terminal velocity (i.e. the velocity of the loop when/if the acceleration goes to zero) of the loop when the upper wire is still in the magnetic field. If the loop never reaches terminal velocity, explain how this comes to be.

QUANTUM MECHANICS (QM)

1. Consider the eigenfunctions $\psi_n(x)$ of a harmonic oscillator with mass m in a potential $V(x) = \frac{1}{2}m\omega^2 x^2$.

(a) Write down an expression for the possible bound-state energies E_n along with the possible values of the quantum number n.

(b) Using the fact that $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$ where $a_+\psi_n(x) = \sqrt{n+1}\ \psi_{n+1}(x)$ and $a_-\psi_n(x) = \sqrt{n}\ \psi_{n-1}(x)$, calculate the expectation value $\langle V(x)\rangle$ for a state $\psi_n(x)$ with energy E_n.

(c) The ground-state wavefunction has the form $\psi_0(x) = Ae^{-bx^2}$. Determine the value of b.

2. An electron is in the spin state $\chi = A\begin{pmatrix} 3i \\ -4i \end{pmatrix}$ in a basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to the state with $S_z = \frac{1}{2}\hbar$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to the state with $S_z = -\frac{1}{2}\hbar$.

(a) Determine the normalization constant A.

(b) Using the matrix operators $S_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $S_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, calculate the expectation values $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$.

(c) If the value of the spin angular momentum along the x-direction is measured for this state, what is the probability P_x^+ that it is equal to $+\frac{1}{2}\hbar$?

3. The electron in a hydrogen atom occupies the combined spin and position state,

$$\psi = R_{21}(r)\left[\sqrt{1/3}\ Y_{00}(\theta,\phi)\ \chi_+ + \sqrt{2/3}\ Y_{11}(\theta,\phi)\ \chi_-\right]$$

where χ_+ corresponds to $S_z = \hbar/2$ and χ_- corresponds to $S_z = -\hbar/2$.

(a) If you measure the orbital angular momentum squared L^2 what values might you get and what is the probability of each?

(b) Same for the z-component of angular momentum L_z.

(c) Same for the spin angular momentum squared S^2.

(d) Same for the z-component of spin angular momentum S_z.

(e) Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$ be the *total* angular momentum. If you measure J^2 what values might you get?