University of Toledo, Department of Physics and Astronomy

Ph.D. Qualifying Exam

Fall 2019
September 28

Instructions:

- **Do not** write your name on your exam. On every sheet of paper that you turn in, put your assigned letter code at the upper left corner and a label identifying the problem (such CM1, EM2, QM3) at the upper right corner.
- Work 2 out of 3 problems on each subject. Staple together solutions for each subject and turn them in separately. Indicate clearly which problems are to be graded and which are to be omitted.
- Begin each problem on a new sheet of paper.

CLASSICAL MECHANICS (CM):

1. A rod of mass m and length l is suspended from one end. The other end is attached to a horizontal massless spring with spring constant k. Using the small-angle approximation determine the oscillation frequency of the rod.

2. A particle of mass m hangs from a spring with spring constant k. A second particle of mass m is suspended from the first particle by a spring which also has spring constant k. Ignoring gravity (since this only shifts the point of equilibrium) determine the normal mode frequencies.

3. A projectile of mass m is launched towards the North pole with speed v at an angle ϕ from the horizontal from a point in the northern Hemisphere with latitude λ. Assuming that the angular velocity of the Earth is given by ω, and that the trajectory is short enough that the Earth can be taken to be flat and neglecting air resistance, determine an expression for the Eastward deflection of the projectile when it lands. Hint: for simplicity neglect the contribution to the Coriolis force from the vertical component of the projectile's velocity.

ELECTRICITY AND MAGNETISM (E&M):

1. Imagine the reflection of a light wave incident onto a sheet of glass, where the x-z plane defines the surface of the glass and the polarization vector of the wave is

$$\hat{n} = 1/2\hat{x} + 1/2\hat{z} + 1/\sqrt{(}2)\hat{y}$$

(the polarization vector points in the direction of the E-field). For E-fields parallel to the plane of the glass, the electric field of the reflected light is given by

$$E_R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_I$$

where E_R is the electric field of the reflected wave and E_I is the electric field of the incident light wave. For E-fields perpendicular to the glass, the reflected wave is given by

$$E_R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) E_I$$

where the variables have the same meaning. Furthermore, we define

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}, \beta = \frac{\mu_1 n_2}{\mu_2 n_1}$$

where n_1 and n_2 are the index of refraction for the air and glass, respectively, and μ_1 and μ_2 are the permeability of the air and gas, respectively. The angles are θ_T, the angle of transmitted wave, and θ_I, the angle of the incident wave. Both of these are measured with respect to a normal vector extending from the glass's surface.

Calculate the polarization vector of the reflected wave. Let the angle of incidence be 45 degrees for the incident wave, the index of refraction of air and glass are 1 and $\sqrt{2} \approx 1.41$, respectively, and assume μ does not change. You need to use Snell's law to start this: $sin(\theta_T)/sin(\theta_I) = n_1/n_2$.

2. Optical tweezers can be used to manipulate microscopic objects using the intense electric field created by a focused light beam. In this problem, you will show that the force on a small dielectric particle in the electric and magnetic fields of the light beam is given by:

$$\vec{F} \propto \vec{\nabla} E^2$$

a. determine the net force on a dipole with a polarization $\vec{P} = q\vec{d} = q(\overrightarrow{x_1} - \overrightarrow{x_2})$ assuming an electric field $\vec{E}$ and magnetic field $\vec{B}$. Also assume that the size of the particle is much smaller than the wavelength of light. To do this, write the net Lorentz force as a function of $\vec{P}$, $\vec{E}$ and $\vec{B}$, and then use the equation $\vec{P} = \alpha\vec{E}$ to write it in terms of α $(\alpha = \epsilon_0 \chi_e)$, $\vec{E}$ and $\vec{B}$.

b. use the equality

$$(\vec{E} \cdot \vec{\nabla})\vec{E} = \vec{\nabla}\left(\frac{1}{2}E^2\right) - \vec{E} \times (\vec{\nabla} \times \vec{E})$$

and Maxwell's equation for the curl of $\vec{E}$ to write the force in terms of $\vec{\nabla} E^2$ and the time derivative of the Poynting vector. Since the flux does not change, the time derivative of the Poynting term is zero.

c. demonstrate that the dielectric object will be pushed to the peak of the electric field in the center of the tweezers.

3. Imagine a cosmic ray moving through a magnetic field in an interstellar cloud.

a. The kinetic energy of the cosmic ray, which is a proton (mass = 1.67×10^{-27} kg), is 10^7 eV (1 eV = 1.6×10^{-12} erg). What is the velocity of the particle? (Ignore relativistic effects.)

b. The magnetic field is 30 μG, where 10^4 Gauss = 1 Tesla. Given that the charge of the electron is 1.6×10^{-19} Coulombs, calculate the cyclotron frequency of the proton.

c. Assuming that the direction of the particle is rotated by an angle θ relative to the magnetic field vector, what is the radius of the cyclotron orbits?

d. Write the equation for the position of the particle as it spirals through the cloud. Adopt a starting point of $\overrightarrow{X_0}$ at $t = 0$. Make a drawing of this motion.

e. Assuming the particle is moving through a cloud of length l. Write the total path length of the particle trajectory as a function of θ, the particle velocity v, and the length of the cloud l. (If $\theta = 0$ degrees, then the path length is simply the length of the cloud l.) What angle gives the highest path length? Calculate that path length for a cloud of length 3×10^{18} cm if $\theta = 45^\circ$.

QUANTUM MECHANICS (QM):

1. Based on QM, explain why the electron binding energy in He atom is greater than that in H and Li atoms? In the order of magnitude, what is the difference in these binding energies?

2. The wave function

   $$\psi = C[2e^{5i\phi} - e^{-2i\phi}]$$

   describes a plane rotator (ϕ is the angle of rotation). An experiment is conducted to determine the angular momentum *L* of the rotator.
   a) What possible values can one find for the measured angular momentum?
   b) What are the probabilities of finding these values?
   c) Determine the value of constant *C*.

3. Consider a system that is initially in the state

   $$\psi(\theta,\varphi) = A[Y_{1,-1}(\theta,\varphi) + 2Y_{10}(\theta,\varphi) + 3Y_{11}(\theta,\varphi)]$$

   where *A* is a normalization constant.
   (a) Find *A*.
   (b) Find the expectation value of the angular momentum in that state.
   (c) Find the expectation value of the projection of angular momentum in that state.