University of Toledo, Department of Physics and Astronomy

Ph.D. Qualifying Exam

Fall 2020
October 10

Instructions:

- **Do not** write your name on your exam; put your chosen letter on every sheet of paper that you turn in.
- Work 2 out of 3 problems in each category.
- Begin each problem on a new sheet of paper.
- Be sure to state which problems are omitted.

CLASSICAL MECHANICS

1. A yo-yo (see picture) of mass m which is hanging by a string is released from rest.

(a) Using Lagrange's equations determine the acceleration of the yo-yo.

(b) What is the speed of the yo-yo once it has fallen a distance h from its starting point?

(You can treat the yo-yo as a solid disk with radius R with a string wrapped around its circumference.)

2. An object of mass m is suspended by a massless rod of length l from the rim of a horizontal disk of radius R which rotates about a vertical axis passing through its center with angular velocity ω. The rod is constrained by a frictionless suspension to swing in a radial direction only (e.g. no sideways motion).

(a) Using Lagrange's equations, write down a differential equation describing the evolution of the rod angle $\theta(t)$. (b) Assume that the rod has settled down so that $d\theta/dt = d^2\theta/dt^2 = 0$, and $\theta = \text{constant} = \theta_0$. Write down an equation which relates the rod angle θ_0 to the angular velocity ω.

(c) This equation is not easy to solve analytically. However, it can be solved in the limit $\omega \ll 1$ and $\theta_0 \ll 1$. Write down an expression for θ_0 in this limit.

3. Two rods of mass m and length l are each suspended from the ceiling at one end, while their other ends are joined by a horizontal spring with spring constant k.

(a) Determine the normal mode frequencies of the system in the small-angle approximation.

(b) Draw a picture indicating the motion for each of the normal modes.

ELECTROMAGNETISM

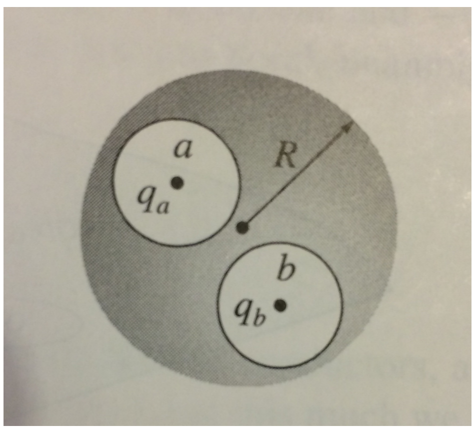
1. Two spherical cavities of radii a and b are hollowed out from the interior of a (neutral) conducting sphere of radius R. At the center of each cavity a point charge is placed - call these charges q_a and q_b.

(a) Find the charge surface densities on the surfaces of each of the two cavities, σ_a and σ_b, and the charge surface density on the outer surface of the sphere, σ_R.

(b) What is the field outside the conductor?

(c) What are the forces on q_a and q_b?

(d) Which of these answers would change if a third charge, q_c, were brought near the conductor?



2. Consider a flat plate capacitor with surface charge densities σ and $-\sigma$.

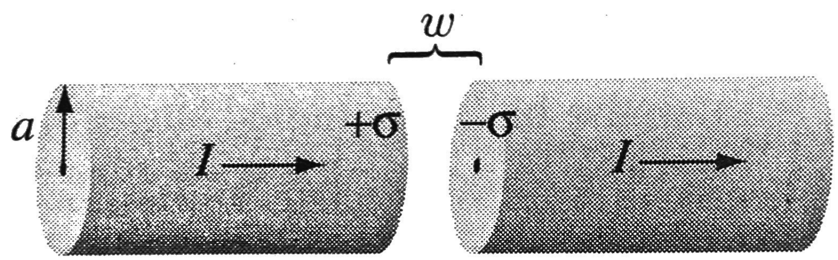
(a) Calculate the vectors of electric field $\vec{E}$ and electric induction $\vec{D}$ inside the capacitor in a vacuum.

(b) Now we fill the capacitor with water assuming that it is disconnected from a power source. The water dielectric constant is $\epsilon = 80$. What is the electric field in the capacitor?

(c) What is the polarization $\vec{P}$ in the water?

(d) By what factor is the capacitance increased when we add water?

3. Imagine a fat wire with radius a that carries a constant current, I, uniformly distributed over its cross section. It is suddenly cut, leading to a narrow gap in the wire of width $w << a$ as shown in the Figure. This gap forms a parallel-plate capacitor. Assuming that the current remains almost constant during a certain time t, positive and negative charges build up on opposing sides of the gap.



(a) Find the electric and magnetic fields in the gap as a function of distance from the axis and the time t.

(b) Find the time dependent energy density u_{em} and the Poynting vector $\vec{S}$ in the gap as function of the distance from the axis of the wire. Note especially the direction of $\vec{S}$. Ignore possible edge effects when the distance is equal to a.

(c) Using the results from part b, determine the time derivative of the total energy in the gap. Compare the change in energy with the flow of energy into the gap predicted by the Poynting vector.

QUANTUM MECHANICS

1. A particle in 1D is prepared in a bound state characterized by a wave function

$$\psi(x) = \begin{cases} Cx^{1/2}(a-x)^{1/2} & 0 \le x \le a \\ 0 & x < 0;\ x > a \end{cases} .$$

   a) Find the normalization constant C.
   b) What is the probability for finding the particle in the region of $[0, a/4]$?
   c) What is the average position $\langle x \rangle$ in this state?
   d) What is the uncertainty of position Δx in this state?

2. Consider physical observables A and B for a two-state quantum system. Operator $\hat{A}$, representing observable A, has two orthonormal eigenstates $|u_1\rangle$ and $|u_2\rangle$, with distinct eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two orthonormal eigenstates $|v_1\rangle$ and $|v_2\rangle$, with distinct eigenvalue b_1 and b_2, respectively. The eigenstates are related by

$$|u_1\rangle = \frac{1}{\sqrt{5}}\left(|v_1\rangle + 2|v_2\rangle\right) ,$$
$$|u_2\rangle = \frac{1}{\sqrt{5}}\left(2|v_1\rangle - |v_2\rangle\right) .$$

   a) Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?
   b) If B is now measured, what are the possible outcomes, and what are the probabilities for each outcome?
   c) Right after a measurement of B, in which b_1 is obtained, A is measured again. What is the probability of getting a_2?

3. A hydrogen atom in a $2p$ state with $m = +1$ is described by a wave function

$$\psi_{211} = R_{21}(r)Y_{11}(\theta, \phi) ,$$

   where

$$R_{21}(r) = \frac{1}{a_0^{3/2}}\frac{1}{2\sqrt{6}}(r/a_0)e^{-r/2a_0} ,$$

   in which a_0 is the Bohr radius, and

$$Y_{11}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi} .$$

   a) What are the average values and the uncertainties of energy, squared angular momentum L^2, and the z-component of the angular momentum L_z in this state?
   b) Find the average radius $\langle r \rangle$ and the uncertainty of radius Δr in this state.
   c) Find the average potential energy and the average kinetic energy.