Quantum Mechanics Section

Possibly useful info:

Energy for the n=1 level of the Hydrogen atom is -13.6 eV.

$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(0)\rangle$

Problem 1

Consider a state function immediately following the colon which is a superposition of two lowest eigenfunctions of the infinite square well whose boundaries are at $x = -a$ and $x = a$ ($V(x) = 0 \quad$ between $-a < x < a$):

$$\Psi = \frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right) + \frac{2}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right)$$

(a) Normalize this wave function in the space $-a <= x <= a$.

(b) Include the appropriate time factor in each term of the wave function and find the probability density function, $\rho(x,t)$.

(c) Using the normalized wave function in (a), calculate the expectation value of the kinetic energy.

Problem 2

Consider a system whose state $|\Psi(t)\rangle$ and two observables A and B are given by:

$$|\Psi(t)\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} -i \\ 2 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & i & 1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 0 \end{bmatrix}$$

(a) Do A and B commute? Include any mathematical reasoning to support your answer.

(b) Measuring A first and then B immediately afterwards, find the probability of obtaining a value of -1 for A and a value of 3 for B.

(c) Now, measuring B first then A immediately afterwards, find the probability of obtaining 3 for B and -1 for A. Compare this result with the probability obtained in (b).

Problem 3

At time $t = 0$, the wave function for a hydrogen atom is:

$$\Psi(r,0) = \frac{1}{\sqrt{10}}(2\Psi_{100} + \Psi_{210} + \sqrt{2}\Psi_{211} + \sqrt{3}\Psi_{21-1})$$

Where the subscripts are the quantum numbers n, l, m. Ignore spin and radiative transitions.

(a) What is the expectation value of energy for this system in units of eV?

(b) What is the probability of finding the system with $l = 1$ and $m = +1$ as a function of time?

(c) What is the probability of finding an electron within $\alpha = 10^{-10}$ cm of a proton at time $t = 0$.

$$|R_{10}|^2 = \frac{4}{a^3}e^{-2r/a}, \quad |R_{21}|^2 = \frac{r^2}{24a^5}e^{-r/2a}, \quad a = 5.29 \times 10^{-9}\,\text{cm.} \;<- \textit{Bohr Orbit}$$

Use the following approximation since $r <= \alpha << a$:

$$e^{-2r/a} \approx 1 - \frac{2r}{a}, \quad e^{-r/2a} \approx 1 - \frac{r}{2a}$$

(d) How does this wave function evolve with time? i.e. what is $\Psi(r,t)$?

Classical Mechanics

1. Two identical springs are each attached to a separate mass M and a distinct wall with a spring between them as shown in Figure 1. The two springs and the connecting one have spring constants κ and κ_{12} respectively. The displacements from equilibrium are denoted as x_1 and x_2. For small oscillations of the two masses to occur find their normal frequencies and modes. Depict the modes with arrows in a figure.

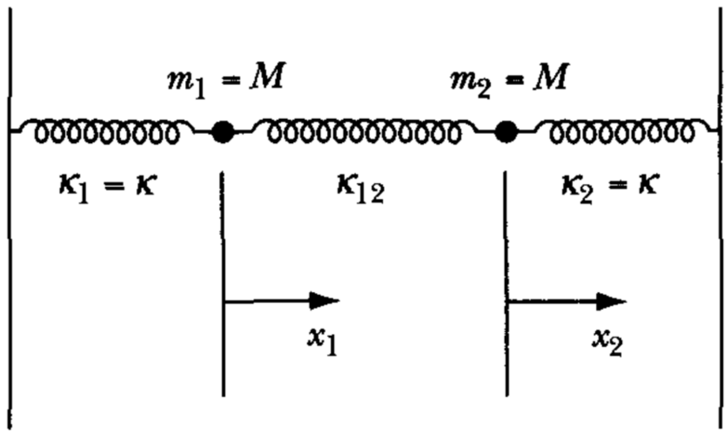


Figure 1.

2. A rigid body consists of a massless rod of length *2b*, each endpoint of which is attached to a mass *m*. Its center of mass is fixed at the origin. The rod makes a fixed angle θ with the positive *z* axis and rotates with angular velocity, $\boldsymbol{\omega}$. Find the angular momentum and torque as function of time in terms of unit vectors along the Cartesian axes and given quantities. Ignore gravity.

3. Mass m_1 starts from rest and slides down an inclined plane from a height y_1 as shown in Figure 2. It then slides a horizontal distance x_1 on a flat table surface having a coefficient of kinetic friction μ. It then sticks to a mass m_2. The two masses together fall to the ground through a height y_2 and move a horizontal distance x_2. The plane makes an angle θ with the horizontal direction. The magnitude of the acceleration due to gravity is g. Find μ in terms of the given quantities.

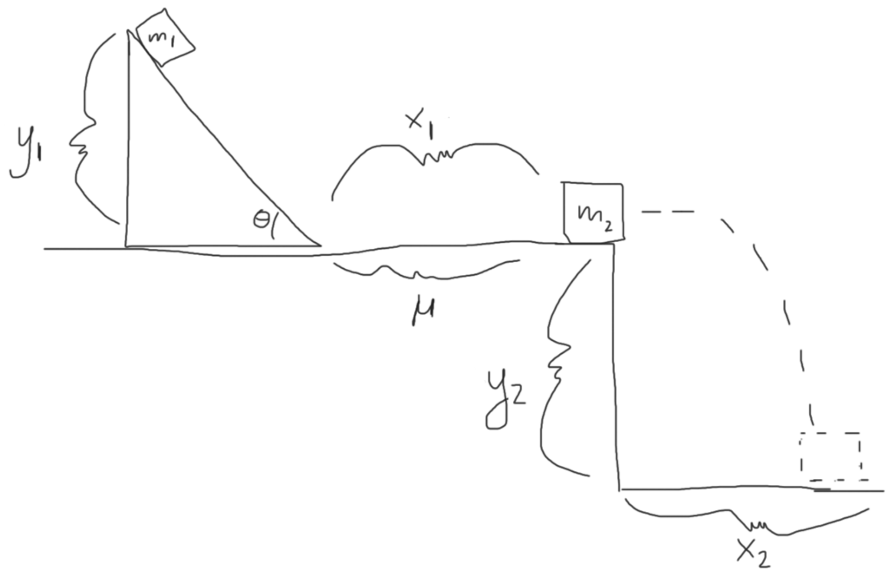
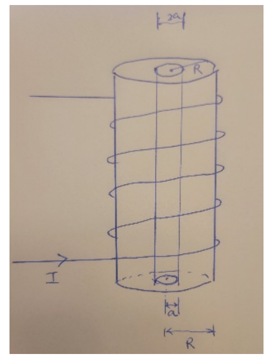


Figure 2.

E&M Section:

Problem 1: A very long solenoid with radius R, n turns per unit length, is carrying a current I. Co-axial with the solenoid is a very long cylindrical (non-conducting) shell inside the solenoid at radius a ($a<R$), carrying a charge Q over a total length of L, and is free to rotate along the axis. Find:

1) Both the electrical field and magnetic field at position r inside the solenoid ($r < R$)
2) Poynting vector at a position r ($<R$)
3) Momentum density of electromagnetic field at a position r ($< R$)
4) Angular momentum of the electromagnetic field inside the solenoid for a section with length L.
5) Angular momentum of the electromagnetic field outside the solenoid in the region ($R < r < 2R$) with a length L.
6) Angular momentum of the inner cylinder (with radius a) when the current I in the solenoid is turned off.



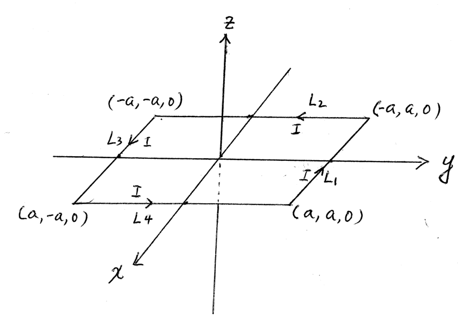
Problem 2: A square loop of wire, with sizes 2a, lies in the x-y plane (centered at the origin) and carries a current I running counterclockwise as viewed from the positive z axis.

(a) What is its magnetic dipole moment?

(b) What is the dipole term of the magnetic vector potential?

(c) What is the (approximate) magnetic field at points far from the origin? (Consider only the dipole term). Express it in spherical coordinate system.

(d) Find, for points on the z axis, the magnetic field when z >> a.



Problem 3: A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in the figure.

(a) Find the magnetic field between the plates and also above and below them.
(b) Find the magnetic force per unit area on the upper plate, including its direction.
(c) At what speed v would the magnetic force balance the electrical force?