University of Toledo, Department of Physics and Astronomy

Ph.D. Qualifying Exam

Fall 2008
September 13

Instructions:

- **Do not** write your name on your exam; put your chosen letter on every sheet of paper that you turn in.
- Work 2 out of 3 problems in each category.
- Begin each problem on a new sheet of paper.
- Be sure to state which problems are omitted.
MECHANICS

1. The ground electronic potential (describing the interaction between two atoms) for an iodine molecule $I_2$ is given by

$$V(r) = A(141e^{-2r/\rho} - 2e^{-r/\rho})$$

Here $r$ is the interatomic distance, $A = 220$ eV, and $\rho = 0.537$ Å. The mass of an iodine atom is 127 atomic mass units (u). [1 eV = $1.60 \times 10^{-19}$ J, 1 u = $1.66 \times 10^{-27}$ kg]

a. Determine the equilibrium separation, $r_0$, between the two atoms (i.e. bond length).

b. Determine the dissociation energy $D_e$, i.e., the minimum energy required to pull apart the two iodine atoms initially in the ground state.

c. Determine the frequency $\nu_0$ for small-amplitude vibrations around the equilibrium.

d. Treating this diatomic molecule as a rigid rotor with interatomic separation $r_0$, derive the moment of inertia of the molecule relative to the center-of-mass.

2. An artificial satellite is observed to have a maximum distance $l_1$ and a minimum distance $l_2$ from the center of the earth during one orbit of the earth. Find its maximum and minimum velocities in terms of $l_1$, $l_2$, the mass of the Earth $M$, and the gravitational constant $G$.

3. A double pendulum consists of 2 rods of mass $m$ each connected to each other and a point of support. The masses are $m_1$ and $m_2$ and the lengths are the same.

a. What is the kinetic energy for the system?

b. What is the potential energy?

c. What is the Lagrangian?

d. What are the equations of motion?
ELECTRICITY & MAGNETISM

1. You find a mysterious sphere of radius $R$ that produces an electric field. To measure the electric field, you shoot a beam of electrons at the sphere and vary the beam energy, direction, and location. From this experiment, you determine (from the deflection of the beam) that the electric field is radially directed with a functional form given by

$$ E = \begin{cases} E_0 (r/R) \hat{r} & (r < R) \\ -E_0 (R/r)^2 \hat{r} & (r > R) \end{cases} $$

a. What is the total charge of your sphere?
b. How is the charge distributed? List:
   i) the location and value of any point charges $q$,
   ii) the location and value of any surface charge densities $\sigma$
   iii) the functional form of any volume charge densities $\rho$.
   Hint: you might want to first sketch the electric field.
c. Assuming that the electron beam is directed radially inward from infinity, find an expression for the minimum required beam energy (initial kinetic energy) for the beam to reach the center of the sphere.
d. What is the kinetic energy of these electrons when they reach the center?

2. Consider a “point” dipole with dipole moment $p = p\hat{z}$ placed at the origin of a grounded conducting spherical shell of radius $a$. Note: a point dipole is one in which the charge separation vanishes ($d \to 0$) while the dipole strength $p = qd$ is held fixed.
a. First ignore the conducting shell, and write down an expression for the electrostatic potential $\Phi(r, \theta)$ at distances large compared to the charge separation but small compared to the radius of the shell (i.e., $d << r << a$).
b. Now include the spherical conducting shell and solve for the electrostatic potential at all positions within the interior of the sphere ($d << r < a$). Hint: In the limit of small $r$, your expression should match the answer for part (a).
c. Find the surface charge density $\sigma$ induced on the surface of the sphere.

3. A long cylindrical permanent magnet of length $L$ and radius $a$ ($a << L$) is magnetized parallel to its axis with a uniform magnetization $\mathbf{M} = M_0 \hat{z}$. Find the magnetic field $\mathbf{B}$
a. near the center of the magnet, $r << a$.
b. at large distances $r >> L$. 
QUANTUM MECHANICS

1. In a basis that comprised of spin-up and spin-down in the z-direction, the angular momentum of a spin 1/2 particle can be described in terms of Pauli matrices:

\[ \vec{S} = \frac{\hbar}{2} \vec{\sigma} \]

where

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

a. If a particle is initially prepared in a spin-up state in the z-direction, what are the possible outcomes if measurements of the y-component of the spin are conducted? What are the corresponding probabilities?
b. What are the average value and the uncertainty of the measurements conducted in a)?
c. A particle is prepared in a spin-up state in the z-direction, as in a). What are the possible outcomes and corresponding probabilities if measurements of spin are conducted along a direction specified by \( \theta=60^\circ, \phi=0^\circ \) in spherical coordinates?

2. This problem consists of three related parts.
   a. Find the energy levels and associated normalized wave functions for a particle in a 1-D box: \( V(x) = 0, \quad 0 \leq x \leq a; \quad V = \infty, \quad x < 0 \text{ or } x > a \).
   b. Find the energy levels and associated normalized wave functions for a particle in a 2-D box: \( V(x,y) = 0, \quad \text{for } 0 \leq x \leq a \text{ and } 0 \leq y \leq b; \quad V = \infty, \text{ elsewhere} \).
   c. For a particle in a 2-D box and under three different scenarios of \( a<b, \ a=b, \ \text{and} \ a>b \), characterize the ground and the first excited energy levels using appropriate quantum numbers and discuss their degeneracy, namely are there multiple states having the same energy?

3. Consider a two-state system described by a Hamiltonian \( H = a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \),

where \( a \) is real constant with a dimension of energy. We prepare it initially in a state \( \psi(t=0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) (called it the “down” state), and let it evolve with time.

a. Without any calculation, do you expect the average energy (also called the expectation value of energy) to stay constant with time, or to vary with time? Explain.
b. Do you expect the average value of a physical observable \( X \), described by operator \( X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), to stay constant with time or to vary with time? Explain or prove your result.
c. Find the wave function at arbitrary time \( t \). What is the probability that the system has “flipped” to the “up” state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) at time \( t \)?
d. Find the expectation values of the energy and the observable \( X \) at time \( t \) to verify your conclusions in a) and b).