Instructions:

- **Do not** write your name on your exam; put your chosen letter on every sheet of paper that you turn in.
- Work 2 out of 3 problems in each category.
- Begin each problem on a new sheet of paper.
- Be sure to state which problems are omitted.
MECHANICS

1. A test particle of mass \( m \) orbits a much more massive object \( (M) \) in a circular fashion a distance \( r \) away.
   a) Derive the potential associated with the gravitational force;
   b) Similarly, derive the potential associated with the ‘fictional’ centrifugal force in terms of angular momentum \( L \);
   c) Sketch the two potentials as a plot of energy versus \( r \) as well as the resulting effective potential;
   d) What is the maximum potential energy of the test particle?

2. When a mass of 5 kg rests on the top of a vertical spring originally 1 m long, the top of the spring is depressed 10 cm. If the mass is now pushed down an additional 20 cm and released, compute how far above the fixed lower end of the spring the mass rises.

3. A layer of dust is formed \( h \) feet thick (\( h \) small compared to the earth's radius) by the fall of meteors reaching the earth from all directions. Show, by considering angular momentum, that the change in the length of the day is approximately \( (5hd/RD) \) of a day, where \( R \) is the radius of the earth, and \( D \) and \( d \) are the densities of the earth and the dust, respectively. *Hint:* The moment of inertia of a sphere about an axis through its center is \( (2/5)MR^2 \).
ELECTRICITY & MAGNETISM

1. A capacitor is created by placing a small conducting sphere of radius $a$ at a distance $d$ ($a \ll d$) above a grounded conducting infinite plane.
   a. Assuming that a charge $Q$ is placed on the sphere, find an expression for the electrostatic potential $\Phi$ as a function of position. Hint: the surface charge density on the sphere is approximately uniform.
   b. What is the capacitance $C$ of this capacitor?

2. A non-conducting dielectric sphere of radius $a$ and permittivity $\varepsilon$ has a surface charge density $\sigma = \sigma_0 \cos \theta$ placed on its surface.
   a. Find the electrostatic potential $\Phi$ both inside and outside the sphere.
   b. What are the electric fields $\mathbf{D}$ and $\mathbf{E}$ inside the sphere?

3. A small circular wire loop of radius $a$, carrying current $I_1$ in the $\phi$-direction, is placed with its center at the coordinate origin and oriented with its axis in the $z$-direction. A second circular loop, also of radius $a$, is oriented parallel to the first, centered on the $z$-axis, and placed a distance $z$ ($a \ll z$) above the first loop.
   a. What is the magnetic moment $\mathbf{m}$ of the 1st current loop?
   b. What is the magnetic field $\mathbf{B}_{12}$ produced by the 1st loop at the center of the 2nd loop?
   c. Suppose the 2nd loop is moving towards the 1st loop with a velocity $\mathbf{v} = -v\hat{z}$. If the 2nd loop has a resistance $R$, what is the current $I_2$ induced in the 2nd loop?
   d. What is the direction of the induced current, $+\phi$ or $-\phi$? Explain your reasoning.
Quantum Problems

1. An operator \( \hat{A} \), representing observable \( A \), has two normalized eigenstates \( \psi_1 \) and \( \psi_2 \), with eigenvalues \( \alpha_1 \) and \( \alpha_2 \), respectively. Operator \( \hat{B} \), representing observable \( B \), has two normalized eigenstates \( \phi_1 \) and \( \phi_2 \), with eigenvalue \( \beta_1 \) and \( \beta_2 \). The eigenstates are related by

\[
\psi_1 = \left( \frac{3\phi_1 + 4\phi_2}{5} \right), \quad \psi_2 = \left( \frac{4\phi_1 - 3\phi_2}{5} \right).
\]

a) Observable \( A \) is measured, and the value \( \alpha_1 \) is obtained. What is the state of the system immediately after this measurement?
b) If \( B \) is now measured, what are the possible results, and what are their probabilities?
c) Right after a measurement of \( B \), in which the value \( \beta_1 \) is obtained, \( A \) is measured again. What is the probability of getting \( \alpha_1 \)?

2. Consider a particle of mass \( m \) confined in a 1-D box: \([V(x)=0, 0 \leq x \leq a; V=\infty, x<0 \text{ or } x>a]\).

a) Find the energy levels and associated normalized wave functions.
b) If the particle is prepared initially \((t=0)\) in state

\[
\psi(x,t=0) = \left( \frac{8}{5a} \right)^{1/2} \left( 1 + \cos \frac{\pi x}{a} \right) \sin \frac{\pi x}{a}
\]

what are the probabilities for finding it in the ground state and the first excited state at \( t=0 \)? What is the average energy at \( t=0 \)?
c) For the particle as prepared in b), find the wave function at time \( t \). What are the probabilities for the particle to be found in the ground state and the first excited state at time \( t \), and what is the average energy at time \( t \)?

3. A hydrogen-like atom in a 3d state with \( m=+2 \) is described by a wave function

\[
\psi = R_{32}(r) Y_{22}(\theta, \phi),
\]

where

\[
R_{32}(r) = \left( \frac{Z}{a_0} \right)^{3/2} \frac{4}{81\sqrt{30}} \left( \frac{Zr}{a_0} \right)^2 \exp(-Zr/3a_0), \quad \text{and} \quad Y_{22} = \frac{1}{4} \frac{15}{2\pi} \sin^2 \theta \ e^{i2\phi}
\]

in which \( a_0 \) is the Bohr radius.

a) Find, from the physical meaning of the quantum numbers, the average values and the uncertainties of the energy, the squared angular momentum \( L^2 \), and the z-component of the angular momentum \( L_z \) in this state.
b) Find the average radius \( \langle r \rangle \) and the uncertainty of radius \( \Delta r \) in this state.
c) Find the average potential energy and the average kinetic energy in this state.

\[\text{[hint: recall } E_n = -\frac{Z^2}{n^2} \left( \frac{e^2}{2a_0} \right)\]