Fall, 2001

M.S. (Applied Mathematics)
Comprehensive Examination in Analysis

Do five (5) questions from **each** *of Parts A and B.*
Indicate on the front of the blue book which problems you wish to have graded.

$\mathbf{R}$ denotes the set of real numbers and $\mathbf{C}$ the set of complex numbers.

Part A. Real Analysis

1. Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Prove that $\lim_{n\to\infty} a_n b_n = AB$.

2. (a) Let (M,d) be a metric space and suppose $f_n : M \to \mathbf{R}$ are continuous functions. Define what it means for $f_n \rightrightarrows f$ uniformly on M.

   (b) Prove: If $f_n \rightrightarrows f$ uniformly on M and each f_n is continuous on M, then so is f.

3. *Abel's Theorem*: Suppose $\sum_{k=0}^{\infty} a_k$ converges and let $f_n(x) = \sum_{k=0}^{n} a_k x^k$. Show that $f_n(x) \rightrightarrows f(x)$ uniformly on $[0,1]$, where $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

4. (a) Give two different but equivalent definitions of compactness for a metric space (M,d).

   (b) Prove that a compact metric space is closed and bounded and give an example to show that the converse statement is not true.

5. Let $f(x)$ be a continuous real-valued function on $[a,b]$ with $f(x) \geq 0$.
   Prove: If there is one point $c \in [a,b]$ with $f(c) > 0$ then

$$\int_a^b f(x)dx > 0.$$

6. Let $f_n(x) = x^n(1-x)$, $g_n(x) = x^n(1-x^n)$, $0 \leq x \leq 1$. Show that $f_n(x) \rightrightarrows 0$ uniformly on $[0,1]$ while $g(x) \to 0$ pointwise on $[0,1]$ but not uniformly.

7. Let $f : M \to \mathbf{R}$ be a continuous function on the compact metric space (M,d).
   Prove that f is uniformly continuous.

8. (a) Define what it means for a subset O of M to be *open*, where (M,d) is a metric space.

   (b) Prove that for any $p \in M$ and $r > 0$, the ball $B(p,r) = \{q \in M : d(p,q) < r\}$ is an open subset of M.

Part B. Complex Analysis

1. (a) Prove that if $f(z)$ and $\overline{f(z)}$ are analytic in a domain $D \subseteq \mathbf{C}$ then $f(z)$ is constant in D. (*Hint:* Cauchy-Riemann equations)

   (b) Prove that if $f(z)$ is analytic in a domain $D \subseteq \mathbf{C}$ and $|f(z)|$ is constant in D, then $f(z)$ is constant in D.

2. (a) Verify that the function $u(x, y) = 2x(1 - y)$ is harmonic in $\mathbf{R}^2$ and find a harmonic conjugate $v(x, y)$.

   (b) Suppose $u(x, y)$ and $v(x, y)$ are conjugate harmonic functions on a domain $D \subseteq \mathbf{R}^2$. If $U(x, y) = e^{u(x,y)} \cos v(x, y)$ and $V(x, y) = e^{u(x,y)} \sin v(x, y)$, show that $U(x, y)$ and $V(x, y)$ are also conjugate harmonic.

3. (a) Evaluate the contour integral

$$\int_C \frac{dz}{\bar{z}}$$

   where C is the upper half of the circle $|z| = 1$ from $z = 1$ to $z = -1$.

   (b) Let C be the line segment from $z = i$ to $z = 1$. *Without* evaluating the integral directly show that

$$\left| \int_C \frac{dz}{\bar{z}} \right| \leq 2$$

4. Evaluate

$$\oint_C \frac{\sin^2 z}{(z - \pi/6)^3} dz$$

   if C is the circle $|z| = 1$ traced once counterclockwise.

5. (a) State Liouville's Theorem.

   (b) Suppose a non-constant function $f(z)$ is such that, for two constants $a > 0$ and $b > 0$, $f(z) = f(z + a)$ and $f(z) = f(z + bi)$ for all $z \in \mathbf{C}$. (Such a function is said to be *doubly periodic*.) Prove that $f(z)$ is not analytic in the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$.

6. Compute *all* possible Laurent series for $f(z) = \frac{1}{z^2 - 4z + 3}$ at $z = 1$. State explicitly the domain of convergence of each series.

7. Use residues to evaluate the contour integral

$$\oint_C z(3z + 1)e^{2/z} dz$$

   where $C = \{z : |z| = 1\}$ traced once counterclockwise.

8. Use residues to evaluate the improper integral

$$\int_0^\infty \frac{1}{x^4 + 1} dz.$$