Sept. '94 Examiners: P. Hewitt, D.A. White

1. *Laurent Expansions.* Suppose that $f(z)$ is holomorphic in the annulus: $\{z \in \mathbf{C} : 0 \le R_1 < |z-a| < R_2 \le \infty\}$.

   (a) Use Cauchy's Integral Formula to show that

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

   where

   $$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}}\, dt$$

   and $C = \{t : |t-a| = r\}$ for some r, $R_1 < r < R_2$. Show also that the series converges absolutely inside the annulus.

   (b) Assume further that $R_1 = 0$ and, for some $\alpha > 0$, and $M > 0$,

   $$\max_{0\le\theta\le 2\pi} |f(a + re^{i\theta})| \le M r^{-\alpha} \text{ for all } r,\, 0 < r < R_2.$$

   Show that $a_{-n} = 0$ if $n > \alpha$.

2. *Argument Principle.* Suppose that f is meromorphic in a simply connected domain and C is a simple closed rectifiable curve oriented counterclockwise in that domain. Let a_j, $1 \le j \le J$ be all the zeroes of f and b_k, $1 \le k \le K$ all the poles of f inside C, with zeroes counted according to their multiplicity and poles according to their order. Assume also that no zeroes or poles of f lie on C. Interpret $\int_C \frac{f'(z)}{f(z)}\, dz$ and $\int_C z\frac{f'(z)}{f(z)}\, dz$ in terms of the zeroes and poles. Prove all your assertions.

3. *Harmonic Conjugates.*

   (a) Suppose that $u(x,y)$ is an harmonic function defined on a domain D. Show that for every $(a,b) \in D$, there is a function $v(x,y)$ defined and harmonic in a neighborhood of (a,b) which is an "harmonic conjugate" of u in the sense that $f(x+iy) = u(x,y) + iv(x,y)$ defines an holomorphic function.

(b) Show, by specific example that the harmonic conjugate need not be defined on the whole of the original domain D. (Find D and u harmonic on D so that there is no v defined on all D so that f as defined above is holomorphic.)

4. *Montel's Theorem.* A collection $\mathcal{F}$ of functions defined on a common domain D is said to be *normal* if every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on every compact subset of D. State and prove Montel's Theorem, which gives necessary and sufficient conditions under which a collection of functions, all holomorphic on a common domain D, is normal.