Ph.D. Qualifying EXAM
DIFFERENTIAL EQUATIONS
SPRING 2001

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This exam has two parts, ordinary differential equations and partial differential equations. Do all the problems.

Part I: Ordinary Differential Equations

1. Given the non-autonomous system as below. Show that the paths followed by particles emitted at (x_0, y_0) at $t = s$ are the same regardless of the value of s. Why is this so?

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad .\frac{dy}{dt} = \frac{y}{1+t}.$$

2. Consider the system

$$\dot{x} = x, \quad \dot{y} = -y + x^2.$$

(a) Determine the stability and type of the rest point (0,0).
(b) Find a function $\phi(x, y)$ such that all solutions are level curves $\phi(x, y) = c$.

3. Let $y_1(x), y_2(x)$ be two linearly independent solutions of the equation $y'' + p(x)y' + q(x)y = 0$, where $p(x), q(x)$ are continuous functions on $(-\infty, \infty)$. Prove that all the roots of $y_1(x), y_2(x)$ are simple and that between every two roots of $y_1(x)$ there is exactly one root of $y_2(x)$.

Part II: Partial Differential Equations

1. Given the initial value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \quad \text{on} \quad t > 0; \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x); \end{aligned}$$

make the change of variables $\xi = x + ct,\ \zeta = x - ct$ and solve the equation and its initial conditions.

2. Given the second order PDE

$$U_{xx} - 3U_{yy} + 2U_x - U_y + U = 0; \quad U = U(x,y)$$

identify the type of equation and transform it into a canonical form consistent with its type as identified above.

3. Consider the Laplace equation $\Delta u = u_{xx} + u_{yy}$ on the square $0 \leq x \leq \pi,\ 0 \leq y \leq \pi$ with the boundary conditions

$$u(0,y) = u(\pi,y) = 0, \quad u(x,0) = 0, \quad u(x,\pi) = f(x).$$

Use the method of separation of variables to find a series solution and decide the coefficients of the series.