Spring 2002 Ph.D Qualifying Exam in Real Analysis

Time 3 hours, closed book, no notes. Answer three questions from each of the parts A & B.

Part A

1. Prove Egoroff's Theorem namely: Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ (a.e.). Show that for any $\epsilon > 0$, there is a measurable set E with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on $X \backslash E$.

2. Let X be a complete metric space. Suppose $\mathcal{F}$ is a family of continuous functions on X with the property that for each $x \in X$, there is a constant M_x such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Show that there exists a non-empty open set $O \subset X$ and a constant M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and $x \in O$.

3. Let f_n be a sequence of continuous real-valued functions on the closed interval $[a, b]$ with $f_n(x) \leq f_{n+1}(x)$, and with $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in [a, b]$. Prove that if f is continuous on $[a, b]$, then the convergence is uniform.

4. Suppose $\sum_{k=0}^{\infty} a_k$ converges. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}$.

   (a) Show that both series converge uniformly on $[-\rho, \rho]$ for any ρ, $0 < \rho < 1$.

   (b) Show that $f'(x) = g(x)$.

   (c) Show that $\lim_{x\to 1^-} f(x) = \sum_{k=0}^{\infty} a_k$.

5. Let X be a metric space with metric ρ. Show that X is totally bounded if and only if every sequence $\{x_n\}$ in X has a Cauchy subsequence.

6. Let $\phi_n(x) = \sqrt{n} e^{-n^2|x|}$ and let $f(x) = \sum_{n=0}^{\infty} \phi_n(x - r_n)$ where $\{r_n\}$ is an enumeration of rational numbers in $\mathbb{R}$. Show:

   (a) $f(x) \in L^1(\mathbb{R}, m)$ and compute $\int_{\mathbb{R}} f(x)dx$

   (b) $f(x)$ is unbounded in any open interval (a, b).

Part B

1. Show that

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n dx = \lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}dx = 1.$$

2. (a) State the Stone-Weierstrass theorem.

   (b) If f is a continuous function on $[0,1]$ such that $\int_0^1 x^n f(x)dx = 0$ for $n = 1, 3, 5, \cdots$, then show that $f(x) = 0$ for all $x \in [0,1]$.

   (c) Show that the algebra generated by the set $\{1, x^2\}$ is dense in $\mathcal{C}([0,1])$ but fails to be dense in $\mathcal{C}([-1,1])$.

3. Suppose that f, a function defined on an open interval (a,b), satisfies the intermediate value theorem i.e., if f assumes the values y_1, y_2, it assumes all values between y_1, y_2. Show that if f is not continuous, it assumes some value infinitely often.

4. Let $f(x)$ be a continuous real-valued function on a compact metric space (X,ρ). Show that f is uniformly continuous.

5. Let $K(x,y)$ be continuous on the closed unit square $[0,1]$x$[0,1]$ in $\mathbb{R}^2$. For $f \in \mathcal{C}([0,1])$, we define

$$g(x) = T(f)(x) = \int_0^1 K(x,y)f(y)dy.$$

   Prove that if $\{f_n\}$ is a sequence in $\mathcal{C}([0,1])$ with $\|f_n\| \le M$ for all n, there is a subsequence $\{n_k\}$ such that the sequence $\{g_{n_k}\}$ converges uniformly.

6. Let (X,ρ) be a metric space and $A \subset X$ be a non-empty subset. Let $\rho(x,A) = \inf\{\rho(x,z) \mid z \in A\}$, the distance from x to A.

   (a) Show that $\rho(x,A)$ is uniformly continuous on X. In fact $|\rho(x,A) - \rho(y,A)| \le \rho(x,y)$.

   (b) Now show that $\rho(x,A) = 0$ if and only if $x \in \overline{A}$.